is a Hamiltonian cycle C of $T(G)$.

Subcase 2'.2.6 A section of C_1 is $v_1, \cdots, v_3, \cdots, v_4$, where the vertices not written are all vertices of S_1. Then the sequence of vertices

$$w, v_0, v_1, \cdots, v_3, \cdots, v_4, v_2, u, \cdots, v_5, \cdots, w$$

is a Hamiltonian cycle C of $T(G)$.

By above discussion, it is easy to see that $T(G)$ has a Hamiltonian cycle when G has n cut vertices, and so the theorem follows. □

Corollary 2.3 *For any tree G of order not less than 2, the sufficient and necessary condition of $T(G) \in H$ is that G is a path.*

Proof By Theorem 1.3, $T(G) \in P$, and by Theorem 2.2, it is easy to see that the conclusion is true. □

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关 于 完 美 全 图 的 Hamilton- 性

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摘　要

设 $G(V,E)$ 是一个简单图, 而 $V(T(G)) = V(G) \cup E(G), E(T(G)) = \{yz | y, z$ 相邻或相关, $y, z \in V(G) \cup E(G)\}$. 则称 $T(G)$ 为 $G(V,E)$ 的全图; 若对 G 的每一导出子图 H, 有 $\chi(H) = \omega(H)$, 则称 G 是完美的. 其中 $\chi(H), \omega(H)$ 分别表示 H 的色数和团数. 本文给出了完美全图是 Hamilton 图的充分必要条件.

Journal of Mathematical Research & Exposition
Vol.16, No.1, 41–46, Feb. 1996

On the Hamiltonity of Perfect Total Graphs *

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Abstract Let $G(V,E)$ be a simple graph, and $V(T(G)) = V(G) \cup E(G)$, $E(T(G)) = \{yz | y$ is adjacent or incident to $z, yz \in V(G) \cup E(G)\}$. Let $\chi(H)$ and $\omega(H)$ be the chromatic number and the number of cliques of H, respectively. Then $T(G)$ is called a total graph of G. G is called perfect if $\chi(H) = \omega(H)$ for each induced subgraph H of G. We give a necessary and sufficient condition for a perfect total graph to hamiltonian.

Keywords perfect total graph, Hamiltonity.

Classification AMS(1991)05C15/CCL O157.5

1 Introduction

Definition 1.1 *Let $G(V,E)$ be a simple graph and*

$$V(T(G)) = V(G) \cup E(G),$$
$$E(T(G)) = \{yz | y \text{ is adjacent or incident to } z, yz \in V(G) \cup E(G)\}.$$

Then $T(G)$ is called a total graph of $G, E(G)$ is called derivative vertex set.

Definition 1.2[1] *Let $\chi(G)$ and $\omega(G)$ be the chromatic number and the number of cliques of G, respectively; $G[S]$ be an induced subgraph of G for $S \subseteq V(G), G$ is called perfect if $\chi(G[S]) = \omega(G([S])$ for any $S \subseteq V(G)$, and it is simply represented by $G \in F$.*

Theorem 1.3[2] *For any simple graph $G, T(G) \in F$, if and only if each block of G includes at most three vertices. $G(V,E)$ is called a hamiltonian graph or we say $G(V,E)$ have hamiltonicity if graph $G(V,E)$ has hamiltonian cycle; it is simply represented by $G - H$.*

We refer to [1],[2] for notations and terminologies not explained here. The graphs that we consider here are simple.

2 The Main Results

Lemma 2.1 *For total graph K of order p,*

$$T(K_2), T(K_3) \in H.$$

*Received Apr.7, 1994. The project supported by the NNSF of China, GPNSF and RMNSF.

The conclusion is trivial. The proof is omitted here.

Theorem 2.2 *For any graph $G, T(G) \in P$, the necessary and sufficient condition of $T(G) \in H$ is that any cut vertex of G is incident to at most two K_2-blocks.*

Proof First we prove the necessary condition.

We will prove it by contradiction. If $v_0 \in V(G)$ and it is a cut vertex of G. Then suppose v_0 is incident to three K_2-blocks $G[\{v_0, v_i\}](i = 1, 2, 3)$, where $v_i \neq v_j(i \neq j, i, j = 1, 2, 3)$. Since $T(G) \in H$, it is easy to see whether v_1, v_2, v_3 are cut vertices of G or not, there will be the following conclusion: There are just two internally- disjoint paths between v_0 and $v_i(i = 1, 2, 3)$.

Suppose that C is a Hamiltonian cycle of $T(G)$, and in cycle C.

$$d_C(v_0) = d_C(v_i) = 2(i = 1, 2, 3).$$

Because v_0 is a cut vertex of G, by Lemma 1 and $v_1v_2, v_2v_3, v_3v_1 \in E(T(G))$, we get v_1, v_2 and v_3 are all incident to v_0 in cycle C. But by hypothesis $v_i \neq v_j(i \neq j, j = 1, 2, 3)$, this is impossible.

We can similary prove that it is impossible for v_0 to be incident to more than three K_2-blocks. So the necessary is true.

Now we prove the sufficient condition by induction on the number of cut vertices n of G.

When $n = 0$, by Theorem 1.3, G is K_2 or K_3. From Lemma 2.1 the conclusion is true.

When $n = 1$, suppose v_0 is the cut vertex. By Theorem 1.3 there are the following three cases.

Case 1 There is no K_2-block in G.

From theorem 1.3, we know each block of G is K_3 at the cut vertex v_0, the vertices of K_3-block are denoted (according to blocks) as $1, 2; 3, 4; \cdots; 2k-1, 2k$; where k is the number of blocks of K_3 in graph G. The derivative vertices are represented by $0 \cdot i(i = 1, 2, \cdots, 2k)$ and $i \cdot (i + 1)(i = 1, 2, \cdots, 2k - 1)$. Where $0 \cdot i$ represents those from the edges between the vertices v_0 and $i(i = 1, 2, \cdots, 2k)$ in $T(G)$; $1 \cdot (i + 1)$ represents those from the edges between vertices v_i and $i + 1(i = 1, 3, \cdots, 2k - 1)$ in $T(G)$. Then the sequence of vertices

$$v_0, 0 \cdot 1, 1, 1 \cdot 2, 2, 0 \cdot 2, 0 \cdot 3, 3, 3 \cdot 4, 4, 0 \cdot 4, \cdots, 0 \cdot (2k - 1), 2k - 1,$$

$$(2k - 1) \cdot (2k), 2k, 0 \cdot (2k), v_0$$

is a Hamiltonian cycle of $T(G)$.

Case 2 G has just only one K_2-block.

First we give K_3-block the similar notation as case 1, the other vertices of K_2-block are represented by $2k + 1$. They have the same meaning as case 1. Then the sequence of vertices

$$v_0, 2k + 1, 0 \cdot (2k + 1), 0, \cdot 1, 1, 1 \cdot 2, 2, \cdots, 0 \cdot (2k - 1), 2k - 1,$$

$$(2k - 1) \cdot (2k), 2k, 0 \cdot (2k), v_0$$

is a Hamiltonian cycle of $T(G)$.

Case 3 G has two K_2-blocks.

Suppose there are kK_3-block in G.

Subcase 3.1 When $k = 0$. The other two vertices of K_3 are represented by 1,2. Then the sequence of vertices

$$v_0, 1, 0\cdot 1, 0\cdot 2, 2, v_0$$

is a Hamiltonian cycle of $T(G)$.

Subcase 3.2 When $k > 0. K_3$ has its vertices the same notation as case 1, the other vertices of two K_2-blocks are represented by $2k+1, 2k+2$, respaceively. Then the sequence of vertices

$$v_0, 2k+1, 0\cdot(2k+1), 0\cdot 1, 1, \cdots, 2k, 0\cdot(2k), 0\cdot(2k+2), 2k+1, v_0$$

is a Hamiltonian cycle of $T(G)$.

By above discussion the conclusion is true for $n = 1$.

Suppose when the number of cut vertices is less than n, it is true that $T(G) \in H$. We now prove that $T(G) \in H$ for the number n of cut vertices.

From the property of G, and since G is a limited graph, G must have a cut vertex v_0 which is adjacent to at most two vertices, and let v_1 be a cut vertex adjacent to it. We consider the following two cases.

Case 1' v_0v_1 is a cut edge of G.

Let G_1 be an induced subgraph of G with vertices of the branch including v_1 of $G - v_0$ and vertex v_0; G_2 be an induced subgraph of G with vertices of the branch incoluding v_0 of $G - v_1$ and vertex v_1. Then G_2 has one cut vertex v_0, G_1 has $n-1$ cut vertices. By the induction hypothesis,

$$T(G_1), T(G_2) \in H.$$

Let v_2 represent the derivative vertex from edge v_0v_1 of G in $T(G), C_1$ the Hamiltonian cycle of $T(G_i)(i = 1, 2)$. Since

$$d_{T(G_1)}(v_0) = d_{T(G_2)}(v_1) = 2,$$

edges v_0v_1, v_0v_2 are in the cycle C_1, v_1v_0, v_1v_2 are in the cycle C_2. We construct a cycle C based on C_1, C_2 and let C satisfy

$$V(C) = V(C_1) \cup V(C_2),$$
$$E(C) = [E(C_1) \cup E(C_2)] \backslash \{v_0v_2, v_1v_2\}.$$

Then the cycle C is a Hamiltonian cycle of $T(G)$.

Case 2' v_0v_1 is not a cut edge of G.

By the conditions given in the theorem, G must have a vertex v_2 such that $v_0v_2, v_1v_2 \in E(G)$. Otherwise, v_0v_1 will be a cut edge of G. This is a contradiction.

Subcase 2'.1 If v_2 is not a cut vertex of G.

Let G_1 represent the induced subgraph of G with the vertex v_0 and vertices which belong to the branch of $G - v_0$ incoluding v_1, G_2 the induced subgraph of G with the vertex v_0 and vertices of $V(G)\backslash V(G - v_0); v_3, v_4, v_5$ the derivative vertices from edges v_0v_1, v_1v_2, v_2v_0 in $T(G)$; and

$$S_0 = \{v_0, v_1, v_2, v_3, v_4, v_5\};$$
$$S_1 = V(T(G_1))\backslash S_0;$$
$$S_2 = V(T(G_2))\backslash\{v_0\}.$$

Then G_2 has at most one cut vertex G_1 has $n-1$ cut vertices. By the induction hypothesis, $T(G_i) \in H(i = 1, 2)$. From $n = 0$ or $n = 1$, we can suppose that $v_0, u, \cdots, w, v_0$ is a Hamiltonian cycle C_2 of $T(G_2)$, where $u, w \in V(T(G_2))$, the other vertices (not writeen) are all vertices of $S_2\backslash\{u, w\}$, and $v_5u, wv_3 \in E(T(G))$.

Let C_1 represent a hamiltonian cycle of $T(G_1)$. We consider that

$$d_{T(G_1)}(v_1) = 4(i = 0, 2, 5)$$

and the vertices v_0, v_2, v_5 are not adjacent to any one of S_1 in $T(G_1)$, then we have the following six subcases.

Subcase 2'.1.1 A section of C_1 is $v_3, \cdots, v_1$, where the vertices not written are all vertices of S_1. Then the sequence of vertices $w, v_3, \cdots, v_1, v_4, v_2, v_5, v_0, u, \cdots, w$ is a Hamiltonian cycle C of $T(G)$.

Subcase 2'.1.2 A section of C_1 is $v_3, \cdots, v_4$, where the vertices not written are all vertices of S_1. Then the sequence of vertices $w, v_3, \cdots, v_4, v_1, v_2, v_5, v_0, u, \cdots, w$ is a Hamiltonian cycle C of $T(G)$.

Subcase 2'.1.3 A section of C_1 is $v_1, \cdots, v_4$, where the vertices not written are all vertices of S_1. Then the sequence of vertices $w, v_3, v_1, \cdots, v_4, v_2, v_5, v_0, u, \cdots, w$ is a Hamiltonian cycle C of $T(G)$.

Subcase 2'.1.4 A section of C_1 is $v_3, \cdots, v_1, \cdots, v_4$, where the vertices not written are all vertices of S_1. Then the sequence of vertices $w, v_3, \cdots, v_1, \cdots, v_4, v_2, v_5, v_0, u, \cdots, w$ is a Hamiltonian cycle C of $T(G)$.

Subcase 2'.1.5 A section of C_1 is $v_3, \cdots, v_4, \cdots, v_1$, where the vertices not written are all vertices of S_1. Then the sequence of vertices $w, v_3, \cdots, v_4, \cdots, v_1, v_2, v_5, v_0, u, \cdots, w$ is a Hamiltonian cycle C of $T(G)$.

Subcase 2'.1.6 A section of C_1 is $v_1, \cdots, v_3, \cdots, v_4$, where the vertices not written are all vertices of S_1. Then the sequence of vertices $w, v_0, v_1, \cdots, v_3, \cdots, v_4, v_2, v_5, u, \cdots, w$ is a Hamiltonian cycle C of $T(G)$.

By above discussions, it is easy to see that the conclusion is true when v_2 is not a cut vertex of G.

Subcase 2'.2 Each cut vertex of G is at least adjacent to two cut vertices.

As G is a limited graph. It's easy to prove by contradiction that G must have two adjacent cut vertices v_0, v_2 which have only one common adjacent cut vertex v_1.

Let G_1 represent the induced subgraph of G with vertices v_0, v_2 and vertices of the branch of $G - \{v_0, v_2\}$ including vertex v_1, G_2 the induced subgraph of G with vertex v_1 and vertices of the branches of $G - v_1$ including vertex v_0 and $v_2; v_3, v_4, v_5$ have the same meaning as subcase 2.1, and

$$S_0 = \{v_0, v_1, v_2, v_3, v_4\};$$
$$S_i = V(T(G_i)) \backslash (S_0 \cup \{v_5\}), \quad (i = 1, 2).$$

Suppose u is a derivative vertex from one of edges incident to vertex v_2; w is a derivetive vertex from one of edges incident to vertex $v_0, u, w \overline{\in} \{v_3, v_4, v_5\}$; and the sequence of vertices $u, \cdots, v_5, \cdots, w$ represents a hamiltonian path including all vertices of S_2 and v_5 from u to w. Evidently, this path must exist.

Because G_2 has two cut vertices v_0, v_2 and G_1 has $n-2$ cut vertices, $T(G_1)$ must have a hamiltonian cycle C_1 by the hypothesis.

We consider C_1 as the following six subcases similarly to subcase 2'.1.

Subcase 2'.2.1 A section of C_1 is $v_1, \cdots, v_3$, where the vertices not written are all vertices of S_1. Then the sequence of vertices

$$w, v_0, v_3, \cdots, v_1, v_4, v_2, u, \cdots, v_5, \cdots, w$$

is a Hamiltonian cycle C of $T(G)$.

Subcase 2'.2.2 A section of C_1 is $v_1, \cdots, v_4$, where the vertices not written are all vertices of S_1. Then the sequence of vertices

$$w, v_0, v_3, v_1, \cdots, v_4, v_2, u, \cdots, v_5, \cdots, w$$

is a Hamiltonian cycle C of $T(G)$.

Subcase 2'.2.3 A section of C_1 is $v_3, \cdots, v_4$, where the vertices not written are all vertices of S_1. Then the sequence of vertices

$$w, v_0, v_1, v_3, \cdots, v_4, v_2, u, \cdots, v_5, \cdots, w$$

is a Hamiltonian cycle C of $T(G)$.

Subcase 2'.2.4 A section of C_1 is $v_3, \cdots, v_1, \cdots, v_4$, where the vertices not written are all vertices of S_1. Then the sequence of vertices

$$w, v_0, v_3, \cdots, v_1, \cdots, v_4, v_2, u, \cdots, v_5, \cdots, w$$

is a Hamiltonian cycle C of $T(G)$.

Subcase 2'.2.5 A section of C_1 is $v_3, \cdots, v_4, \cdots, v_1$, where the vertices not written are all vertices of S_1. Then the sequence of vertices

$$w, v_0, v_3, \cdots, v_4, \cdots, v_1, v_2, u, \cdots, v_5, \cdots, w$$

is a Hamiltonian cycle C of $T(G)$.

Subcase 2'.2.6 A section of C_1 is $v_1, \cdots, v_3, \cdots, v_4$, where the vertices not written are all vertices of S_1. Then the sequence of vertices

$$w, v_0, v_1, \cdots, v_3, \cdots, v_4, v_2, u, \cdots, v_5, \cdots, w$$

is a Hamiltonian cycle C of $T(G)$.

By above discussion, it is easy to see that $T(G)$ has a Hamiltonian cycle when G has n cut vertices, and so the theorem follows. □

Corollary 2.3 *For any tree G of order not less than 2, the sufficient and necessary condition of $T(G) \in H$ is that G is a path.*

Proof By Theorem 1.3, $T(G) \in P$, and by Theorem 2.2, it is easy to see that the conclusion is true. □

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