If $\delta_k = -1$,

$$x_k v_{i_1,\cdots,i_n} = i_k v_{i_1,\cdots,i_k-1,\cdots i_n}, 0 \le i_k \le \pi - 1,$$
$$\theta_k v_{i_1,\cdots,i_{k-1},\pi-1,i_{k+1},\cdots i_n} = \mu v_{i_1,\cdots,i_{k-1},0,i_{k+1},\cdots,i_n},$$
$$\theta_k v_{i_1,\cdots,i_n} = v_{i_1,\cdots,i_{k-1},i_k+1,\cdots,i_n}, 0 \le i_k < \pi - 1,$$

where

$$v_{i_1,\cdots,-1,\cdots,i_n} = v_{i_1,\cdots,\pi-1,i_n}, v_{i_1,\cdots,\pi,\cdots i_n} = v_{i_1,\cdots,0,\cdots,i_n}.$$

Then $V(\delta, \lambda, \mu)$ *is a finite dimensional irreducible module over* $A_n(K)$.

(b) $V(\delta, \lambda, \mu) \simeq V(\delta', \lambda', \mu')$, *if and only if one of the following conditions holds for each* $0 \le k \le n$:

(i) $\delta_k = \delta' k, \lambda_k - \lambda'_k \in I, \mu_k = \mu'_k$.

(ii) $\delta_k = 1, \delta'_k = -1, \mu_k = \mu'_k = 0$.

(iii) $\delta_k = -1, \delta'_k = 1, \mu_k = \mu'_k = 0$.

The proof of this theorem is similar to that of theorem 8.

Theorem 11 *Let* V *be a finite dimensional irreducible* $A_n(K)$*-module . Then* V *is isomorphic to one of the following modules:* $V(\delta, \lambda, \mu)$ *where* $\lambda = (\lambda_1, \cdots, \lambda_n), \mu = (\mu_1, \cdots, \mu_n), \in K^n$, *and* $\delta = (\delta_1, \cdots, \delta_n)$ *with* $\delta_i = 1, 0$ or -1 *are as in Theorem 10.*

The proof of this theorem is similar to that of theorem 9.

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Weyl 代 数 的 表 示

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摘　要

设 K 是一个域. 证明了, 若 ch$K$=0, 那么 n-th Weyl 代数 $A_n(k)$ 没有有限维表示. 还给出了 $A_n(k)$ 的不可约 Harish-Chandra 模的分类. 当 K 是一个特征非零的代数闭域时, 给出了有限维不可约 $A_n(K)$- 模的分类.

Journal of Mathematical Research & Exposition
Vol.16, No.2, 159-166, May 1996

Representations of Weyl Algebras *

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Abstract Let K be a field. In this paper, we prove that if $\mathrm{ch}\,K = 0$, then the n-th Weyl algebra $A_n(K)$ has no finite dimensional representation. And the classification of irreducible (and indecomposable) Harish-Chandra modules over the n-th Weyl algebra $A_n(K)$ is given. If K is an algebraically closed field of characteristic $\neq 0$, then the classification of finite dimensional irreducible $A_n(K)$-modules is got.

Keywords n-th weylalgebra $A_n(k)$, irreducible modules, Harcsh- Chandra modules, finite dimensional representation.

Classification AMS(1991) 17B/CCL O175.3

1. Introduction

Let R be a commutative ring with identity. The n-th Weyl algebra $A_n(R)$ over R is the associative algebra with identity, generated by the $2n$ elements $x_1, x_2, \cdots, x_n, \theta_1, \theta_2, \cdots, \theta_n$ subject to the relations:

$$x_i x_j - x_j x_i = 0 = \theta_i\theta_j - \theta_j\theta_i, \quad x_i\theta_j - \theta_j x_i = \delta_{i,j}. \tag{1}$$

It is clear that $A_n(R)$ has an R-basis $\{\theta_1^{i_1}\cdots\theta_n^{i_n}x_1^{j_1}\cdots x_n^{j_n} | i_1, \cdots, i_n, j_1, \cdots, j_n \in \mathbf{Z}_+\}$. See [5].

In the case that R is a field K, the n-th Weyl algebra $A_n(R)$ has been much studied. About the module structure of $A_n(R)$, J. T. Stafford did a series of thorough studies.

In the present paper, we prove that if $\mathrm{ch}K = 0$, then the n-th Weyl algebra $A_n(K)$ has no finite dimensional representation. And the classification of irreducible (and indecomposable) Harish-Chandra modules over the n-th Weyl algebra $A_n(K)$ is given. If K is an algebraically closed field of characteristic $\neq 0$, then the classification of finite dimensional irreducible $A_n(K)$-modules is got.

2. Representations of $A_1(K)$ with $\mathrm{ch}K = 0$

*Received Nov.7, 1993.

In this section we assume that the characteristic of K is 0.

Set $h_1 = \theta_1 x_1$. We have $[h_1, \theta_1^i x_1^j] = (i-j)\theta_1^i x_1^j$. So h_1 is an ad-semisimple element of $A_1(K)$.

Theorem 1 *The 1-th Weyl algebra $A_1(K)$ has no finite dimensional representation.*

Proof Suppose that V is a finite dimensional $A_1(K)$-module. Choose an arbitrary basis $\{v_1, v_2, \cdots, v_m,\}$ of V. Assume that

$$x_1 \begin{pmatrix} v_1 \\ v_2 \\ \cdots \\ v_m \end{pmatrix} = B \begin{pmatrix} v_1 \\ v_2 \\ \cdots \\ v_m \end{pmatrix}; \quad \theta_1 \begin{pmatrix} v_1 \\ v_2 \\ \cdots \\ v_m \end{pmatrix} = C \begin{pmatrix} v_1 \\ v_2 \\ \cdots \\ v_m \end{pmatrix},$$

where $B = (b_{i,j})$, $C = (c_{i,j})$ are $m \times m$ matrices over K.

From the relation $x_1\theta_1 - \theta_1 x_1 = 1$ it follows that $CB - BC = I_m$. Comparing the traces on both sides, it forces that $m = 0$, contrary to the fact $m > 0$. Therefore $A_1(K)$ has no finite dimensional representation. □

Corollary 1 *The n-th Weyl algebra $A_n(K)$ has no finite dimensional representation.* □

Definition 1 *A module V over $A_1(K)$ is called a Harish-Chandra module over $(A_1(K), h_1)$ if*

(a) $V = \bigoplus_{\lambda \in K} V_\lambda$, *where* $V_\lambda = \{v \in V | h_1 v = \lambda v\}$,

(b) $\dim V_\lambda < \infty$ *for all* $\lambda \in K$.

Theorem 2 (a) *For any $\lambda \in K$, let $V(\lambda) = \bigoplus_{i \in \mathbf{Z}} K v_i$, and*

$$h_1 v_i = (\lambda - i) v_i, \quad x_1 v_i = v_{i+1}, \quad \theta_1 v_i = (\lambda - i + 1) v_{i-1}.$$

Then $V(\lambda)$ is a $A_1(K)$-module.

(b) *For any $\lambda \in K$, let $\bar{V}(\lambda) = \bigoplus_{i \in \mathbf{Z}} K v_i$, and*

$$h_1 v_i = (\lambda - i) v_i, \quad \theta_1 v_i = v_{i-1}, \quad x_1 v_i = (\lambda - i) v_{i+1}.$$

Then $\bar{V}(\lambda)$ is a $A_1(K)$-module.

(c) $V(\lambda) \simeq V(\lambda_1)$, *if and only if,* $\lambda - \lambda_1 \in \mathbf{Z}$.

(d) $\bar{V}(\lambda) \simeq \bar{V}(\lambda_1)$, *if and only if,* $\lambda - \lambda_1 \in \mathbf{Z}$.

(e) $V(\lambda)$ *is irreducible, if and only if,* $\lambda \in \mathbf{Z}$.

(f) $\bar{V}(\lambda)$ *is irreducible, if and only if,* $\lambda \in \mathbf{Z}$.

(g) $V(\lambda) \simeq \bar{V}(\lambda_1)$, *if and only if,* $\lambda - \lambda_1 \in \mathbf{Z}$, *and* $\lambda \notin \mathbf{Z}$.

Proof (a) and (b) are clear.

(c) "$\Longrightarrow$" is obvious.

"$\Longrightarrow$". Let $\lambda_1 = \lambda + r, r \in \mathbf{Z}$, $V(\lambda) = \bigoplus_{i \in \mathbf{Z}} K v_i$, $V(\lambda_1) = \bigoplus_{i \in \mathbf{Z}} K u_i$. It is easy to check that the following linear mapping

$$\begin{aligned} \phi: \ & V(\lambda) \to V(\lambda_1) \\ & v_i \mapsto u_{i+r} \end{aligned}$$

is an isomorphism between $A_1(K)$-modules $V(\lambda)$ and $V(\lambda_1)$.

(d) is similar to (c).

(e) Suppose that $V(\lambda)$ has a proper nonzero module V. Let $v_i \in V$. We have $v_j \in V$ for all $j \geq i$. Then there exists $r \in \mathbf{Z}_+$ such that $v_{i-r} \notin V$. From the relations $\theta_1^r v_i = (\lambda - i + 1) \cdots (\lambda - i + r) v_{i-r} \in V$, we get that $(\lambda - i + 1) \cdots (\lambda - i + r) = 0$. Therefore $\lambda \in \mathbf{Z}$. On the other hand, $V = \bigoplus_{i \geq 1} K v_i$ is a proper submodule of $V(0)$. Therefore (e) is proved.

(f) is similar to (e).

(g) It is easy to see that there exists no $\lambda \in K$ such that $V(0) \simeq \bar{V}(\lambda)$ and there exists not $\lambda \in K$ such that $\bar{V}(0) \simeq V(\lambda)$. So we can suppose that $\lambda, \lambda_1 \notin \mathbf{Z}$.

"$\Longrightarrow$". From $V(\lambda) \simeq \bar{V}(\lambda_1)$ We can easily get $\lambda - \lambda_1 \in \mathbf{Z}$.

"$\Longleftarrow$". If $\lambda - \lambda_1 \in \mathbf{Z}$, by (c) we can assume that $\lambda = \lambda_1$. Write $V(\lambda) = \bigoplus_{i \in \mathbf{Z}} K v_i$ and $V(\lambda_1) = \bigoplus_{i \in \mathbf{Z}} K u_i$. It is easy to check that the linear mapping

$$\begin{aligned} \phi: \ & V(\lambda) \to \bar{V}(\lambda) \\ & v_0 \mapsto u_0 \\ & v_i \mapsto \lambda \cdots (\lambda - i + 1) u_i, i > 0 \\ & v_{-i} \mapsto (\lambda + 1)^{-1} \cdots (\lambda + i)^{-1} u_{-i}, i > 0 \end{aligned}$$

is an isomorphism from $V(\lambda)$ to $\bar{V}(\lambda)$. □

Let $V(0) = \bigoplus_{i \in \mathbf{Z}} K v_i$ and $\bar{V}(0) = \bigoplus_{i \in \mathbf{Z}} K u_i$. Denote the irreducible submodule $\bigoplus_{i \geq 0} K v_i$ of $V(0)$ by $V'(0)$, and the irreducible quotient module $V(0)/V'(0)$ by $\bar{V}'(0)$. It is easy to show that $\bar{V}'(0) \simeq V = \bigoplus_{i \leq 0} K u_i$ which is the unique proper submodule of $\bar{V}(0)$, and $V'(0) \simeq \bar{V}(0)/V$.

If $\lambda \notin \mathbf{Z}$, set $V'(\lambda) = V(\lambda)$. Therefore we get the following irreducible Harish-Chandra modules:

$$V'(\lambda), \quad \bar{V}'(0), \quad \lambda \in K.$$

Theorem 3 *If V is an irreducible Harish-Chandra module over $(A_1(K), h_1)$, then V is isomorphic to one of the following modules : $V'(\lambda), \bar{V}'(0), \lambda \in K$.*

Proof By definition 1 we know that there exists $\lambda \in K$ such that

$$V = \bigoplus_{i \in \mathbf{Z}} V_{\lambda + i}, \quad V_\lambda \neq 0.$$

If $\lambda \notin \mathbf{Z}$, we get that the actions of x_1 and θ_1 on V are faithful, i.e., $0 = \{v \in V | x_1 v = 0\} = \{v \in V | \theta_1 v = 0\}$. In this case we can easily get $V \simeq V'(\lambda)$.

Suppose $\lambda \in \mathbf{Z}$. We claim that there exists $v \in V_\lambda, v \neq 0$ such that $x_1 v = 0$ or $\theta_1 v = 0$. Otherwise the subspace

$$V^{(0)} = \sum_{i \geq 0} K x_1^i v \oplus \sum_{j > 0} K \theta_1^j v$$

is a submodule of V, so

$$V = \sum_{i \geq 0} K x_1^i v \oplus \sum_{j > 0} K \theta_1^j v.$$

If $\lambda = r \geq 0$, we get that $\theta_1 x_1^{r+1} v = h(x_1^r v) = (\lambda - r)v = 0$. If $\lambda = r < 0$, we get that $\theta_1 x_1 \theta_1^{-r} v = h_1(\theta_1^{-r} v) = (\lambda - r)v = 0$. We know that the assertion is true.

If $x_1^k v = 0$ for certain $k \in \mathbf{Z}$, we replace λ by $\lambda + k$, and v by $x_1^k v$. Assume $x_1^k v \neq 0$. (The case $\theta_1^k v = 0$ is similar). If $x_1 v = 0$, we have $h_1 v = 0$, i.e., $v \in V_0$. In this case we can get $V \simeq \bar{V}'(0)$. If $\theta_1 v = 0$, we have $h_1 v = -v$, i.e., $v \in V_{-1}$. In this case we get $V \simeq V'(0)$. So theorem 3 is shown. □

Now we consider indecomposable Harish-Chandra modules over $(A_1(K), h_1)$.

Lemma 1 *Let $V = \bigoplus_{i\in\mathbf{Z}} V_{\lambda+i}$, $\lambda \in K$ be a Harish-Chandra module over $(A_1(K), h_1)$, where $V_{\lambda+i} = \{v \in V | h_1 v = (\lambda + i)v\}$. Then*

(a) If $\lambda + i \neq 0$, then the action of x_1 on $V_{\lambda+i}$ is faithful.

(b) If $\lambda + i \neq -1$, then the action of θ_1 on $V_{\lambda+i}$ is faithful.

Theorem 4 *Let V be an indecomposable Harish-Chandra modules over $(A_1(K), h_1)$. Then V is isomorphic to one of the following modules: $V(\lambda), \bar{V}(\lambda), V'(0)$ or $\bar{V}'(0)$, where $\lambda \in K$.*

Proof. There exists $\lambda \in K$ such that $V = \bigoplus_{i\in\mathbf{Z}} V_{\lambda+i}$. If $\lambda \notin \mathbf{Z}$, we know that the actions of x_1 and θ_1 on V are faithful. Let $V_\lambda = Kv_0^{(1)} \bigoplus \cdots \bigoplus Kv_0^{(r)}$. It is easy to see that for each $s = 1, \cdots, r$ the subspace

$$V^{(s)} = \sum_{i\geq 0} K x_1^i v_0^{(s)} \oplus \sum_{j>0} K\theta_1^j v_0^{(s)} \simeq V(\lambda) \simeq \bar{V}(\lambda)$$

is a submodule of V. From $\lambda \notin Z$ it follows easily that $V = V^{(1)} \bigoplus \cdots \bigoplus V^{(r)}$. By the indecomposability of V we get that $r = 1$ and $V \simeq V(\lambda)$.

Assume that $\lambda \in \mathbf{Z}$. We choose $v \in V_\lambda$, $v \neq 0$. If $\lambda \geq 0$, by lemma 1, $x_1^\lambda v \neq 0$, $x_1^\lambda v \in V_0$. If $\lambda < 0$, then by lemma 1, $\theta_1^{-\lambda-1} v \neq 0$, $\theta_1^{-\lambda-1} v \in V_{-1}$, then $V_0 \neq 0$ or $V_{-1} \neq 0$.

Assume that $V_0 \neq 0$, we write $V_0 = Kv_0^{(1)} \bigoplus \cdots \bigoplus Kv_0^{(r)}, r \geq 1$. We define elements $v_1^{(s)}$ as follows, when $x_1 v_0^{(s)} \neq 0$, let $v_1^{(s)} = x_1 v_0^{(s)}$; when $x_1 v_0^{(s)} = 0$ and there exists $\tilde{v}_1^{(s)} \in V_1$ such that $\theta_1 \tilde{v}_0^{(s)} = v_0^{(s)}$, let $v_1^{(s)} = \tilde{v}_1^{(s)}$; when $x_1 v_0^{(s)} = 0$ and there exists no $\tilde{v} \in V_1$ such that $\theta_1 \tilde{v} = v_0^{(s)}$, let $v_1^{(s)} = 0$. Set $V^{(s)} = \sum_{i\geq 0} K x_1^i v_1^{(s)} \bigoplus \sum_{j>0} K\theta_1^j v_0^{(s)}$. Then $V^{(s)}$ is a submodule of V. Set $T = \{v \in V_{-1} | \theta_1 v = 0\}$. Choose a suitable subspace $S \in T$ such that $T = S \bigoplus x_1 V_0$. Then $A_1(K)S$ is a submodule of V. Therefore by lemma I and $T = S \bigoplus x_1 V_0$ we have the decomposition

$$V = V^{(1)} \oplus \cdots \oplus V^{(r)} \oplus A_1(K)S$$

of submodules. Hence $V = V^{(1)}$. So $V \simeq \bar{V}(0)$ or $\bar{V}'(0)$.

Similarly we can show that $V \simeq V(0)$ or $V'(0)$ in the case $V_{-1} \neq 0$.

Therefore we complet the proof of the theorem. □

3. Harish-Chandra representations of $A_n(K)$ with $\mathrm{ch}K = 0$

In this section we also assume that the characteristic of K is 0.

In $A_n(K)$, set $h_i = \theta_i x_i, 1 \le i \le n$, and $H = Kh_1 \oplus \cdots \oplus Kh_n$. We have

$$[h_k, \theta_1^{i_1} \cdots \theta_n^{i_n} x_1^{j_1} \cdots x_n^{j_n}] = (i_k - j_k)\theta_1^{i_1} \cdots \theta_n^{i_n} x_1^{j_1} \cdots x_n^{j_n}.$$

So $h_i (i = 1, \cdots, n)$ are ad-semisimple elements of $A_n(K)$.

Definition 2 *A module V over $A_n(K)$ is called a Harish-Chandra module over $(A_n(K), H)$ if*

(a) $V = \bigoplus_{\lambda \in K} V_\lambda$, where $\lambda = (\lambda_1, \cdots, \lambda_n) \in K^n, V_\lambda = \{v \in V | h_i v = \lambda_i v, i = 1, \cdots, n\}$,

(b) $\dim V_\lambda < \infty$ for all $\lambda \in K^n$.

Theorem 5 *Let $\lambda = (\lambda_1, \cdots, \lambda_n) \in K^n$.*

(a) Let $V(\lambda_1^{(\delta_1)}, \cdots, \lambda_n^{(\delta_n)}) = \bigoplus_{(i_1, \cdots, i_n) \in \mathbf{Z}^n} Kv_{i_1, \cdots, i_n}$, where $\delta_i = 0$ or 1. Define an action of $A_n(K)$ on $V(\lambda_1^{\delta_1}, \cdots, \lambda_n^{\delta_n})$ as follows

If $\delta_k = 0, x_k v_{i_1, \cdots, i_n} = v_{i_1, \cdots, i_k+1, \cdots, i_n}, \theta_k v_{i_1, \cdots, i_n} = (\lambda_k - i_k) v_{i_1, \cdots, i_k-1, \cdots, i_n}$;

If $\delta_k = 1, x_k v_{i_1, \cdots, i_n} = (\lambda_k - i_k + 1) v_{i_1, \cdots, i_k+1, \cdots, i_n}, \theta_k v_{i_1, \cdots, i_n} = v_{i_1, \cdots, i_k-1, \cdots, i_n}$.

Then $V(\lambda_1^{\delta_1}, \cdots, \lambda_n^{\delta_n})$ is an $A_n(K)$-module.

(b) $V(\lambda_1^{(\delta_1)}, \cdots, \lambda_n^{(\delta_n)})$ is irreducible, if and only if, one of $\lambda_1, \cdots, \lambda_n$ is an integer.

(c) $V(\lambda_1^{(\delta_1)}, \cdots, \lambda_n^{(\delta_n)}) \simeq V(\mu_1^{(\delta'_1)}, \cdots, \mu_n^{(\delta'_n)})$, if and only if, the following two conditions hold

(1) $\lambda_i - \mu_i \in \mathbf{Z}, \forall i$ with $1 \le i \le n$,

(2) $\delta_i = \delta'_i$, for $\lambda_i \in \mathbf{Z}$.

The proof of this theorem is similar to that of theorem 2. We omit the details.

We denote the unique irreducible submodule of $V(\lambda_1^{(\delta_1)}, \cdots, \lambda_n^{(\delta_n)})$ by $V'(\lambda_1^{(\delta_1)}, \cdots, \lambda_n^{(\delta_n)})$.

Theorem 6 *If V is an irreducible Harish-Chandra module over $(A_n(K), H)$, then V is isomorphic to one of the following modules : $V'(\lambda_1^{(\delta_1)}, \cdots, \lambda_n^{(\delta_n)})$, where $(\lambda_1, \cdots, \lambda_n) \in K^n, \delta_i = 0$ or 1.*

The proof of this theorem is similar to that of theorem 3. We omit the details.

Theorem 7 *If V is an indecomposable Harish-Chandra module over $(A_n(K), H)$, then V is isomorphic to a submodule of one of the following modules: $V(\lambda_1^{(\delta_1)}, \cdots, \lambda_n^{(\delta_n)})$, where $(\lambda_1, \cdots, \lambda_n) \in K^n, \delta_i = 0$ or 1.*

The proof of this theorem is very direct and similar to that of theorem 4, we omit the detail.

4. Representations of $A_1(K)$

Where K is an algebraically closed field with ch$K > 0$.

In this section we assume that K is an algebraically closed field with ch$K = \pi > 0$. Let $I = \{0, 1, \cdots, \pi - 1\} \subset K$.

Theorem 8 *Let $\lambda, \lambda', \mu, \mu' \in K$.*

(a) If $\lambda \notin I, \mu \ne 0$, let $V(\lambda, \mu) = \bigoplus_{i \in I} Kv_i$. Define an action of $A_1(K)$ on $V(\lambda, \mu)$ as follows

$$x_1 v_i = v_{i+1}, \quad 0 \le i < \pi - 1,$$

$$
\begin{aligned}
&x_1 v_{\pi-1} = \mu v_0,\\
&\theta_1 v_i = (\lambda - i + 1)v_{i-1}, \quad 0 < i \le \pi - 1,\\
&\theta_1 v_0 = \frac{(\lambda+1)}{\mu} v_{\pi-1},
\end{aligned}
$$

where $v_{-1} = v_{\pi-1}, v_\pi = v_0$. Then $V(\lambda,\mu)$ is a finite dimensional irreducible module over $A_1(K)$.

(b) Let $V(0,\mu) = \bigoplus_{i\in I} K v_i$. Define an action of $A_1(K)$ on $V(0,\mu)$ as follows

$$
\begin{aligned}
&\theta_1 v_i = -i v_{i-1}, \quad 0 \le i \le \pi - 1,\\
&x_1 v_i = v_{i+1}, \quad 0 \le i < \pi - 1,\\
&x_1 v_{\pi-1} = \mu v_0,
\end{aligned}
$$

where $v_{-1} = v_{\pi-1}, v_\pi = v_0$. Then $V(0,\mu)$ is a finite dimensional irreducible module over $A_1(K)$.

(c) Let $\bar V(0, \ \mu) = \bigoplus_{i\in I} K v_i$. Define an action of $A_1(K)$ on $\bar V(0,\mu)$ as follows

$$
\begin{aligned}
&x_1 v_i = i v_{i-1}, \quad 0 \le i \le \pi - 1,\\
&\theta_1 v_i = v_{i-1}, \quad 0 \le 0 < \pi - 1,\\
&\theta_1 v_{\pi-1} = \mu v_0,
\end{aligned}
$$

where $v_{-1} = v_{\pi-1}, v_\pi = v_0$. Then $\bar V(0,\mu)$ is a finite dimensional irreducible module over $A_1(K)$.

(d) $V(\lambda,\mu) \simeq V(\lambda',\mu')$, if and only if, $\lambda - \lambda' \in I, \mu = \mu'$.

(e) $V(0,\mu) \simeq V(0,\mu')$, if and only if, $\mu = \mu'$.

(f) $\bar V(0,\mu) \simeq \bar V(0,\mu')$, if and only if, $\mu = \mu'$.

(g) $V(0,\mu) \simeq \bar V(0,\mu')$, if and only if, $\mu = \mu' = 0$.

The proof of this theorem is conventional and standard, we will not give the detail.

We can easily get the following lemma similar to lemma 1.

Lemma 2 *Let $V = \bigoplus_{i\in I} V_{\lambda+i}$, $\lambda \in K$ be a Harish-Chandra module over $(A_1(K), h_1)$, where $V_{\lambda+i} = \{v \in V | h_1 v = (\lambda+i)v\}$. Then*

(a) If $\lambda + i \ne 0$, the action of x_1 on $V_{\lambda+i}$ is faithful.

(b) If $\lambda + i \ne -1$, the action of θ_1 on $V_{\lambda+i}$ is faithful.

Theorem 9 *Let V be a finite dimensional irreducible module over $A_1(K)$. Then V is isomorphic to one of the following modules: $V(\lambda), V(0,\mu), \bar V(0,\mu)$, where $\lambda, \mu \in K$.*

Proof Since K is algebraically closed and V is finite dimensional, then there exist $\lambda \in K, v \in V$ such that $h_1 v = \lambda v$. So $V = \sum_{i\in \mathbf{Z}_+} K x_1^i v + \sum_{j\in\mathbf{Z}_+} K\theta_1^j v$. Therefore $V = \bigoplus_{i\in I} V_{\lambda+i}$, where $V_{\lambda+i} = \{v \in V | h_1 v = (\lambda+i)v\}$ and $\dim V_{\lambda+i} < \infty$.

(I) Suppose $\lambda \notin I$. In this case, by lemma 2 we know that the actions of x and λ on V are faithful. It is easy to see that h_1, x_1^π and θ_1^π are commutative with each other. There exists a vector $0 \ne v_0 \in V_\lambda$ and nonzero $\mu \in K$ such that $x_1^\pi v_0 = \mu v_0$. So $\bigoplus_{i\in I} K x_1^i v_0$ is a submodule of V. Therefore $V = \bigoplus_{i\in I} K x_1^i v_0$. Set $v_i = x_1^i v_0, 0 \le i < \pi - 1$. We get

$$\begin{aligned}
&\theta_1 v_i = (\lambda - i + 1)v_{i-1}, \quad 0 < i \le \pi - 1,\\
&\theta_1 v_0 = \tfrac{(\lambda-1)}{\mu} v_{\pi-1},\\
&x_1 v_i = v_{i+1}, \quad 0 \le i < \pi - 1,\\
&x_1 v_{\pi-1} = \mu v_0.
\end{aligned}$$

Therefore $V \simeq V(\lambda, \mu)$.

(II) Suppose $\lambda \in I$. By lemma 2, we can assume that $\lambda = 0$. If there exist $0 \ne \bar{v} \in V_0$ such that $x_1 \bar{v} = 0$, set $\bar{V}_0 = \sum_{i\in \mathbf{Z}_+} K\theta_1 \bar{v}$. Then $x_1 \bar{V}_0 = 0, \dim V_0 < \infty$. There exists vector $0 \ne v_0 \in \bar{V}_0 \subset V_0$ and $\mu \in K$ such that $\theta_1 x_1^\pi v_0 = \mu v_0$. So $\bigoplus_{i\in I} K\theta_1^i v_0$ is a submodule of V. Therefore $V = \bigoplus_{i\in I} K\theta_1^i v_0$. Set $v_i = \theta_1^i v_0, 0 \le i < \pi - 1$. We get

$$\begin{aligned}
&x_1 v_i = i v_{i-1}, \quad 0 \le i \le \pi - 1,\\
&\theta_1 v_i = v_{i+1}, \quad 0 \le i < \pi - 1,\\
&\theta v_{\pi-1} = \mu v_0.
\end{aligned}$$

Therefore $V \simeq \bar{V}(0,\ \mu)$.

If the action of x_1 on V is faithful, we can get $V \simeq V(0, \mu)$, where $\mu \in K$ in a way similar to (I). Therefore we have completed the proof. $\square$

5. Representations of $A_n(K)$ where K is an algebraically closed field with ch$K > 0$

In this section we assume that K is an algebraically closed field with ch$K = \pi > 0$. Let $I = \{0, 1, \cdots, \pi - 1\} \subset K$. $I^n = \{(i_1, \cdots, i_n) | i_1, \cdots, i_n \in I\}$.

Theorem 10 *Let $\lambda = (\lambda_1, \cdots, \lambda_n), \lambda' = (\lambda'_1, \cdots, \lambda'_n), \mu = (\mu_1, \cdots, \mu_n), \mu' = (\mu'_1, \cdots, \mu'_n), \in K^n$, and $\delta = (\delta_1, \cdots, \delta_n)$, where $\delta_i = 1, 0$ or -1, $\delta' = (\delta'_1, \cdots, \delta'_n)$, where $\delta'_i = 1, 0$, or -1.*

(a) Assume $\lambda_i \notin I, \mu_i \ne 0$, if $\delta_i = 0$; Assume $\lambda_i = 0$, If $\delta_i = \pm 1$. Let $V(\delta, \lambda, \mu) = \bigoplus_{(i_1,\cdots,i_n)\in I^n} K v_{i_1,\cdots,i_n}$. Define an action of $A_n(K)$ on $V(\delta, \lambda, \mu)$ as follows

If $\delta_k = 0$,

$$\begin{aligned}
&x_k v_{i_1,\cdots,i_n} = v_{i_1,\cdots,i_k+1,\cdots i_n}, 0 \le i_k < \pi - 1,\\
&x_k v_{i_1,\cdots,i_{k-1},\pi-1,i_{k+1},\cdots i_n} = \mu v_{i_1,\cdots,i_{k-1},0,i_{k+1},\cdots i_n},\\
&\theta_k v_{(i_1,\cdots,i_n)} = (\lambda_k - i_k + 1) v_{i_1,\cdots,i_{k-1},i_k-1,\cdots,i_n}, 0 < i_k \le \pi - 1,\\
&\theta_k v_{i_1,\cdots,i_{k-1},0,\cdots,i_n} = \frac{(\lambda_k + 1)}{\mu_k} v_{(i_1,\cdots,i_{k-1},\pi-1,\cdots,i_n)};
\end{aligned}$$

if $\delta_k = 1$,

$$\begin{aligned}
&x_k v_{i_1,\cdots,i_n} = v_{i_1,\cdots,i_k+1,\cdots i_n}, 0 \le i_k < \pi - 1,\\
&x_k v_{i_1,\cdots,i_{k-1},\pi-1,i_{k+1},\cdots i_n} = \mu v_{i_1,\cdots,i_{k-1},0,i_{k+1},\cdots i_n},\\
&\theta_k v_{i_1,\cdots,i_n} = -i_k v_{i_1,\cdots,i_{k-1},i_k-1,\cdots,i_n}, 0 \le i_k \le \pi - 1;
\end{aligned}$$

If $\delta_k = -1$,

$$x_k v_{i_1,\cdots,i_n} = i_k v_{i_1,\cdots,i_k-1,\cdots i_n}, 0 \le i_k \le \pi - 1,$$
$$\theta_k v_{i_1,\cdots,i_{k-1},\pi-1,i_{k+1},\cdots i_n} = \mu v_{i_1,\cdots,i_{k-1},0,i_{k+1},\cdots,i_n},$$
$$\theta_k v_{i_1,\cdots,i_n} = v_{i_1,\cdots,i_{k-1},i_k+1,\cdots,i_n}, 0 \le i_k < \pi - 1,$$

where

$$v_{i_1,\cdots,-1,\cdots,i_n} = v_{i_1,\cdots,\pi-1,i_n}, v_{i_1,\cdots,\pi,\cdots i_n} = v_{i_1,\cdots,0,\cdots,i_n}.$$

Then $V(\delta,\lambda,\mu)$ *is a finite dimensional irreducible module over* $A_n(K)$.

(b) $V(\delta,\lambda,\mu) \simeq V(\delta',\lambda',\mu')$, *if and only if one of the following conditions holds for each* $0 \le k \le n$:

(i) $\delta_k = \delta' k, \lambda_k - \lambda'_k \in I, \mu_k = \mu'_k$.

(ii) $\delta_k = 1, \delta'_k = -1, \mu_k = \mu'_k = 0$.

(iii) $\delta_k = -1, \delta'_k = 1, \mu_k = \mu'_k = 0$.

The proof of this theorem is similar to that of theorem 8.

Theorem 11 *Let* V *be a finite dimensional irreducible* $A_n(K)$*-module . Then* V *is isomorphic to one of the following modules:* $V(\delta,\lambda,\mu)$ *where* $\lambda = (\lambda_1,\cdots,\lambda_n), \mu = (\mu_1,\cdots,\mu_n), \in K^n$, *and* $\delta = (\delta_1,\cdots,\delta_n)$ *with* $\delta_i = 1, 0$ or -1 *are as in Theorem 10.*

The proof of this theorem is similar to that of theorem 9.

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Weyl 代 数 的 表 示

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摘　要

设 K 是一个域. 证明了，若 ch$K=0$, 那么 n-th Weyl 代数 $A_n(k)$ 没有有限维表示. 还给出了 $A_n(k)$ 的不可约 Harish-Chandra 模的分类. 当 K 是一个特征非零的代数闭域时，给出了有限维不可约 $A_n(K)$- 模的分类.