References

[1] A.R.Sourour, *A factorization theorem for matrices* , Linear and Multilinear Algebra, **19**(1986), 141-147.

[2] You Hong and Nan Jizu, *Decomposition of matrices into 2-involutions* , Lin. Algebra. Appl., **186**(1993), 235-253.

[3] Wang Luqun and Zhang Yongzheng, *Two-dimensional group over ϕ-surjective ring*, Acta Mathematic Sinica, Vol.27. No.6, 1984, 860-864.

[4] C.K.Fong and A.R.Sourour, *The group generated by unipotent operators* , Proc. Amer. Math. Soc., **97**(1986), 453-458.

[5] B.R.Mcdonald, *Geometric Algebra over Local Rings* , Pure and Appl Math., Dekker, New York, 1976.

[6] R.C.Thomposon, *Commutators in the special and general linear groups* , Trans. Amer. Math. Soc., **101**(1961), 16-33.

[7] P.Y.Wu, *The operator factorization problem* , Lin. Algebra. Appl., **17**(1989), 35-63.

[8] Wang Luqun and Zhang Yongzheng, *On 2-dimensional linear group with ϕ-full morphism rings* , Kexue Tongbao, Vol.28, No.1, 1983.

ϕ- 满射环上 $\mathrm{GL}_n(R)$ 中元素的三角分解

南 基 洙

(哈尔滨工业大学数学系， 150001)

张 永 正

(东北师范大学数学系，长春 130024)

摘 要

本文结果是：设 A 是 ϕ- 满射环 R 上的非拟纯量可逆 $n\times n$ 矩阵，$\beta_j,\gamma_j(1\le j\le n)$ 是 R 中任意元素，它们满足 $\prod_{j=1}^n\beta_j\gamma_j=\det A$, 则存在 n 阶阵 B 和 C 满足 $PAP^{-1}=BC$, 其中 B 是下三角阵， C 是上三角阵，$P\in\mathrm{GL}_n(R)$. 进一步，可以取 B 使 $\beta_j(1\le j\le n)$ 位于 B 的主对角线上，同时可以取 C 使 $\gamma_j(1\le j\le n)$ 位于 C 的主对角线上.

Journal of Mathematical Research & Exposition
Vol.16, No.2, 173-178, May 1996

The Triangular Factorization for Elements of $\mathrm{GL}_n(R)$ over ϕ-Surjective Rings *

Nan Jizhu
(Harbin Institute of Technology, Harbin 150001)

Zhang Yongzheng
(Dept. of Math. Northeast Normal University, Changchun 130024)

Abstract In this paper, we extend the main theorem in [1]. Our main result is: Let A be a non-near scalar invertible $n \times n$ matrix over ϕ-surjective ring R. Let β_j and $\gamma_j (1 \leq j \leq n)$ be any elements of R such that $\prod_{i=1}^{n} \beta_j \gamma_j = \det A$. Then there exist $n \times n$ matrices B and C such that $PAP^{-1} = BC$, where B is a lower triangular and C is simultaneously upper triangularizable $P \in \mathrm{GL}_n(R)$. Furthermore B and C can be chosen so that the elements in the main diagonal line of B are $\beta_1, \cdots, \beta_n$ and of C are $\gamma_1, \cdots, \gamma_n$.

Keywords ϕ-surjective ring, $\mathrm{GL}_n(R)$.

Classification AMS(1991) 15A23/CCL O151.21

1. Preliminaries

Let R be a commutative ring with 1. $\mathrm{Max}(R)$ is the set of all maximum ideals of R. If $M_t \in \mathrm{Max}(R)$, then λ_t denotes the natural homomorphism of R into R/M_t. $U(R)$ denotes the multiplicative group of unit elements of R. If there exists a subset $\{M_t | t \in T\}$ of $\max(R)$ such that

$$\phi : x \to (\cdots, \lambda_t(x), \cdots)$$

is a surjective ring homomorphism of R into $\prod_{t \in T} R/M_t$ and $\phi(A)$ is also a proper ideal of R for any proper ideal A, then we call R the ϕ-surjective ring (see [3] or [8]).

We know that semilocal ring, derict product of infinite fields and formal power seres ring are ϕ-surjective rings. If R is a ϕ-surjective ring, $x \in R$, then $x \in U(R)$ if and only if $\lambda_t(x) \neq 0, \forall t \in T$(see [3] or [8]).

In the following, R alway denotes a ϕ-surjective ring. $M_n(R)$ denotes the ring of all $n \times n$ matrices over R. $\mathrm{GL}_n(R)$ denotes the group of all invertible $n \times n$ matrices over R. The homomorphism λ_t of R into K_t induces the natural homomorphism λ_t of $M_n(R)$ in $M_n(K_t)$, where $K_t = R/M_t, \forall t \in T$. It is easy to prove that $A \in \mathrm{GL}_n(R)$ if and only

*Received Sep.21, 1993.

if $\lambda_t(A) \in \mathrm{GL}_n(K_t), \forall t \in T$. In this paper $T_{ij}(c), i \neq j$, denotes the matrix whose (i,j)-entry is c and its elements of other positions are the same as the elements of unit matrix I. $E(i,j)$ denotes the matrix which is obtained by exchanging i-row with j-row of I. $D_j(c)$ denotes the matrix multipling i-row by c.

Definition 1 *Let $A \in \mathrm{GL}_n(R)$. If there exists some $t \in T$ such that $\lambda_t(A)$ is a scalar matrix over K_t, i.e., $\lambda_t(A)$ has the following form*

$$\begin{pmatrix} \lambda_t(a_{11}) & & & \\ & \lambda_t(a_{22}) & & \\ & & \ddots & \\ & & & \lambda_t(a_{nn}) \end{pmatrix},$$

$\lambda_t(a_{11}) = \lambda_t(a_{ii}), i = 2, \cdots, n$, then we call A is a near scalar matrix. Otherwise, we call A a non-near scalar matrix

2. Triangular Factorization

Lemma 1 *Let $n \geq 2, A \in \mathrm{GL}_n(R)$. Then A is a non-near scalar matrix if and only if A is similar to the matrix*

$$\begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 1 & a_{22} & \cdots & a_{2n} \\ 0 & a_{32} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Proof $\Rightarrow$ Let A be a non-near scalar matrix, then for $\forall t \in T, \lambda_t(A)$ is not a scalar matrix. By [2], we know that there exists some $B^t \in \mathrm{GL}_n(K_t)$ such that $B^t(\lambda_t A)(B^t)^{-1} = A_0^t$, where

$$A_0^t = \begin{pmatrix} 0 & a_{12}^t & \cdots & a_{1n}^t \\ 1 & a_{22}^t & \cdots & a_{2n}^t \\ 0 & a_{32}^t & \cdots & a_{3n}^t \\ \cdots & \cdots & \cdots & \cdots \\ 0 & a_{n2}^t & \cdots & a_{nn}^t \end{pmatrix}.$$

Because R is a ϕ-surjective ring, there exists some matrix $B \in M_n(R)$ such that $\lambda_t B = B^t, \forall t \in T$. Then we have $B \in \mathrm{GL}_n(R)$ and

$$\lambda_t(BAB^{-1}) = (\lambda_t B)(\lambda_t B^{-1}) = B^t(\lambda_t A)(B^t)^{-1} = A_0^t, \quad \forall t \in T.$$

Let $BAB^{-1} = (c_{ij})$. Then $\lambda_t(c_{21}) = 1 \neq 0, \forall t \in T$. So $c_{21} \in U(R)$. Conjugating BAB^{-1} by $D_2(a_{21}^{-1})$, we can suppose that the element of (2.1)-position of BAB^{-1} is 1. Also conjugating BAB^{-1} by $\prod_{\substack{i=1 \\ i\neq 2}}^{n} T_{i2}(-c_{i1})$, we get the proof of necessary condition.

$\Leftarrow$ By hypothesis of the lemma, there exists $B \in \mathrm{GL}_n(R)$ such that

$$BAB^{-1} = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 1 & a_{22} & \cdots & a_{2n} \\ 0 & a_{32} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix}. \tag{1}$$

If A is a near scalar matrix, then there exists some $t \in T$ such that $\lambda_t A$ is a scalar matrix over K_t. Thus

$$\lambda_t(BAB^{-1}) = (\lambda_t B)(\lambda_t A)(\lambda_t B^{-1}) = \lambda_t A. \tag{2}$$

By equality (1), the element of (2.1)-position of $\lambda_t(BAB^{-1})$ is 1. So $\lambda_t(BAB^{-1})$ is not a scalar matrix. It contradicts equality (2). So A must be a non-near scalar matrix.

Lemma 2 *Suppose* $n \geq 3. A = \begin{pmatrix} \beta_1\gamma_1 & Y \\ X & H \end{pmatrix} \in GL_n(R)$, *where* $H \in M_{n-1}(R), X = (1,0,\cdots,0)'$. *If* $H - \beta_1^{-1}\gamma_1^{-1}XY$ *is a near scalar matrix, then there exists some* $z \in R$ *and* $1 \times (n-1)$ *matrix* $Q = (0,-z,0,\cdots,0)$ *such that* $H - \beta_1^{-1}\gamma_1^{-1}XY_1$ *is a non-near scalar matrix, where* $Y_1 = Y + QH$.

Proof Let

$$H = \begin{pmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}, \quad Y = (y_2, y_3, \cdots, y_n).$$

Then

$$H - \beta_1^{-1}\gamma_1^{-1}XY = \begin{pmatrix} a_{22} - \beta_1^{-1}\gamma_1^{-1}y_2 & a_{32} - \beta_1^{-1}\gamma_1^{-1}y_3 & \cdots & a_{2n} - \beta_1^{-1}\gamma_1^{-1}y_n \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}.$$

Let $T_1 = \{t \in T | \lambda_t(H - \beta_1^{-1}\gamma_1^{-1}XY)$ is a scalar matrix$\}$. Because $H - \beta_1^{-1}\gamma_1^{-1}XY$ is a near scalar matrix, $T_1 \neq \emptyset$. Since R is a ϕ-surjective ring, then there exists some $z \in R$ such that $\lambda_t(z) = 1, \forall t \in T_1$, and $\lambda_t(z) = 0, \forall t \in T/T_1$. Let $Q = (0, z, 0, \cdots, 0), Y_1 = Y + QH$. Then

$$H - \beta_1^{-1}\gamma_1^{-1}XY_1$$
$$= \begin{pmatrix} a_{22} - \beta_1^{-1}\gamma_1^{-1}y_2 + \beta_1^{-1}\gamma_1^{-1}a_{32}z & a_{23} - \beta_1^{-1}\gamma_1^{-1}y_3 + \beta_1^{-1}\gamma_1^{-1}a_{33}z & \cdots & a_{2n} - \beta_1^{-1}\gamma_1^{-1}y_n + \beta_1^{-1}\gamma_1^{-1}a_{3n}z \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

Since $\lambda_t(H - \beta_1^{-1}\gamma_1^{-1}XY)$ is a scalar matrixx, $\forall t \in T_1$, then $\lambda_t(a_{23} - \beta_1^{-1}\gamma_1^{-1}y_3) = 0$ and $\lambda_t(a_{33}) \neq 0, \forall t \in T_1$. Thus $\lambda_t(a_{23} - \beta_1^{-1}\gamma_1^{-1}y_3 + \beta_1^{-1}\gamma_1^{-1}a_{33}z) \neq 0, \forall t \in T_1$. So $\lambda_t(H - \beta_1^{-1}\gamma_1^{-1}XY_1)$ is not a scalar matrix, $\forall t \in T_1$.

Clearly, $\lambda_t(H - \beta_1^{-1}\gamma_1^{-1}XY_1) = \lambda_t(H - \beta_1^{-1}\gamma_1^{-1}XY), \forall t \in T/T_1$. Since $\lambda_t(H - \beta_1^{-1}\gamma_1^{-1}XY)$ is not a scalar matrix, $\forall t \in T/T_1$. So $H - \beta_1^{-1}\gamma_1^{-1}XY_1$ is a non-near scalar matrix.

Theorem 1 *Let A be a non-near scalar inventible $n \times n$ matrix over a ϕ-surjective ring and let β_j and $\gamma_j (1 \le j \le n)$ be elements of R such that $\prod_{j=1}^{n} \beta_j \gamma_j = \det A$. Then there exist $n \times n$ matrices B and C such that $PAP^{-1} = BC$, where B is lower triangularizable and C is simultaneously upper triangularizable, $P \in GL_n(R)$. Furthermore B and C can be chosen so that the elements in main diagonal of B are $\beta_1, \cdots, \beta_n$ and of C are $\gamma_1, \cdots, \gamma_n$.*

Proof We use induction on n. The result is trivially true for $n = 1$. Now we assume that the conclusion of the theorem is true for all square matrices with size less than $n, n \ge 2$, and let A, β_j and γ_j be as in the statement of the theorem. Since A is not a nearscalar matrix, by lemma 1, A is similar to the matrix

$$A_0 = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 1 & a_{22} & \cdots & a_{2n} \\ 0 & a_{32} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Let $P_0 = \begin{pmatrix} 1 & \beta_1\gamma_1 \\ & 1 \end{pmatrix} + I_{n-2}$. Then

$$P_0 A_0 P_0^{-1} = \begin{pmatrix} \beta_1\gamma_1 & b_{12} & \cdots & b_{1n} \\ 1 & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & b_{n2} & \cdots & b_{nn} \end{pmatrix}.$$

So A is similar to the matrix

$$A_1 = \begin{pmatrix} \beta_1\gamma_1 & Y \\ X & T \end{pmatrix},$$

where $X = (1, 0, \cdots, 0)'$.

In the case $n = 2$, we have that X, Y and T are merely elements of R. Suppose $X = x, Y = y$ and $T = t$. Using the fact $\det A_1 = \beta_1\beta_2\gamma_1\gamma_2$ we have

$$\begin{pmatrix} \beta_1\gamma_1 & y \\ x & t \end{pmatrix} = \begin{pmatrix} \beta_1 & 0 \\ x\gamma_1^{-1} & \beta_2 \end{pmatrix} \begin{pmatrix} \gamma_1 & \beta_1^{-1} y \\ 0 & \gamma_2 \end{pmatrix}.$$

This proves the conclusion of the theorem for $n = 2$.

We now assume that $n \ge 3$. If $T - \beta_1^{-1}\gamma_1^{-1}XY$ is not a near scalar matrix, then

$$A_1 = \begin{pmatrix} 1 & \\ \beta_1^{-1}\gamma_1^{-1}X & 1 \end{pmatrix} \begin{pmatrix} \beta_1\gamma_1 & \\ & T - \beta_1^{-1}\gamma_1^{-1}XY \end{pmatrix} \begin{pmatrix} 1 & \beta_1^{-1}\gamma_1^{-1}Y \\ & I \end{pmatrix}. \qquad (\Delta)$$

Obviously $\det A_1 = (\beta_1\gamma_1)\det(T - \beta_1^{-1}\gamma_1^{-1}XY) = \prod_{i=1}^{n} \beta_i\gamma_r$. By hypothesis of induction

$$T - \beta_1^{-1}\gamma_1^{-1}XY = P \begin{pmatrix} \beta_2 & & \\ \vdots & \ddots & \\ * & \cdots & \beta_n \end{pmatrix} \begin{pmatrix} \gamma_2 & \cdots & * \\ & \ddots & \vdots \\ & & \gamma_n \end{pmatrix} P^{-1},$$

then

$$A_1=\begin{pmatrix}1 & \\ \beta_1^{-1}\gamma_1^{-1}X & I\end{pmatrix}\begin{pmatrix}1 & \\ & P\end{pmatrix}\begin{pmatrix}\beta_1 & & & & \\ 0 & \beta_2 & & & \\ 0 & * & \beta_3 & & \\ \vdots & \vdots & & \ddots & \\ 0 & * & \cdots & \cdots & \beta_n\end{pmatrix}\begin{pmatrix}\gamma_1 & 0 & 0 & \cdots & 0\\ & \gamma_2 & * & \cdots & *\\ & & \gamma_3 & & \vdots\\ & & & \ddots & \\ & & & & \gamma_n\end{pmatrix}$$

$$\cdot\begin{pmatrix}1 & \\ & P^{-1}\end{pmatrix}\begin{pmatrix}1 & \beta_1^{-1}\gamma_1^{-1}Y\\ & 1\end{pmatrix}$$

$$=\begin{pmatrix}1 & \\ & P\end{pmatrix}\begin{pmatrix}1 & \\ \beta_1^{-1}\gamma_1^{-1}P^{-1}X & I\end{pmatrix}\begin{pmatrix}\beta_1 & & & \\ 0 & \beta_2 & & \\ \vdots & \vdots & \ddots & \\ 0 & * & \cdots & \beta_n\end{pmatrix}\begin{pmatrix}\gamma_1 & 0 & \cdots & 0\\ & \gamma_2 & \cdots & *\\ & & \ddots & \vdots\\ & & & \gamma_n\end{pmatrix}$$

$$\cdot\begin{pmatrix}1 & \beta_1^{-1}\gamma_1^{-1}YP\\ & 1\end{pmatrix}\begin{pmatrix}1 & \\ & P^{-1}\end{pmatrix}$$

$$=\begin{pmatrix}1 & \\ & P\end{pmatrix}\begin{pmatrix}\beta_1 & & \\ \vdots & \ddots & \\ * & \cdots & \beta_n\end{pmatrix}\begin{pmatrix}\gamma_1 & \cdots & *\\ & \ddots & \vdots\\ & & \gamma_n\end{pmatrix}\begin{pmatrix}1 & \\ & P^{-1}\end{pmatrix}. \qquad (*)$$

If $T-\beta_1^{-1}\gamma_1^{-1}XY$ is a near scalar matrix, then A_1 is similar to the matrix

$$A_2=\begin{pmatrix}\beta_1\gamma_1 & Y_1\\ X & T\end{pmatrix}\begin{pmatrix}1 & -Q\\ & 1\end{pmatrix},$$

where

$$\begin{pmatrix}\beta_1\gamma_1 & Y_1\\ X & T\end{pmatrix}=\begin{pmatrix}1 & Q\\ & I\end{pmatrix}A_1,\quad Q=(0,-z,0,\cdots,0).$$

By Lemma 2, $T-\beta_1^{-1}\gamma_1^{-1}XY_1$ is not a near scalar matrix. By equalities (△) and (*)

$$\begin{aligned}A_2 &= \begin{pmatrix}1 & \\ \beta_1^{-1}\gamma_1^{-1}X & I\end{pmatrix}\begin{pmatrix}\beta_1\gamma_1 & \\ & T-\beta_1^{-1}\gamma_1^{-1}XY_1\end{pmatrix}\begin{pmatrix}1 & \beta_1^{-1}\gamma_1^{-1}Y_1\\ & I\end{pmatrix}\begin{pmatrix}1 & -Q\\ & I\end{pmatrix}\\
&= \begin{pmatrix}1 & \\ & P\end{pmatrix}\begin{pmatrix}\beta_1 & & \\ \vdots & \ddots & \\ * & \cdots & \beta_n\end{pmatrix}\begin{pmatrix}\gamma_1 & \cdots & *\\ & \ddots & \vdots\\ & & \gamma_n\end{pmatrix}\begin{pmatrix}1 & \\ & P^{-1}\end{pmatrix}\begin{pmatrix}1 & -Q\\ & I\end{pmatrix}\\
&= \begin{pmatrix}1 & \\ & P\end{pmatrix}\begin{pmatrix}\beta_1 & & \\ \vdots & \ddots & \\ * & \cdots & \beta_n\end{pmatrix}\begin{pmatrix}\gamma_1 & \cdots & *\\ & \ddots & \vdots\\ & & \gamma_n\end{pmatrix}\begin{pmatrix}1 & -QP\\ & I\end{pmatrix}\begin{pmatrix}1 & \\ & P^{-1}\end{pmatrix}\\
&= \begin{pmatrix}1 & \\ & P\end{pmatrix}\begin{pmatrix}\beta_1 & & \\ \vdots & \ddots & \\ * & \cdots & \beta_n\end{pmatrix}\begin{pmatrix}\gamma_1 & \cdots & *\\ & \ddots & \vdots\\ & & \gamma_n\end{pmatrix}\begin{pmatrix}1 & \\ & P^{-1}\end{pmatrix}.\end{aligned}$$

References

[1] A.R.Sourour, *A factorization theorem for matrices*, Linear and Multilinear Algebra, **19**(1986), 141–147.

[2] You Hong and Nan Jizu, *Decomposition of matrices into 2-involutions*, Lin. Algebra. Appl., **186**(1993), 235–253.

[3] Wang Luqun and Zhang Yongzheng, *Two-dimensional group over ϕ-surjective ring*, Acta Mathematic Sinica, Vol.27. No.6, 1984, 860–864.

[4] C.K.Fong and A.R.Sourour, *The group generated by unipotent operators*, Proc. Amer. Math. Soc., **97**(1986), 453–458.

[5] B.R.Mcdonald, *Geometric Algebra over Local Rings*, Pure and Appl Math., Dekker, New York, 1976.

[6] R.C.Thomposon, *Commutators in the special and general linear groups*, Trans. Amer. Math. Soc., **101**(1961), 16–33.

[7] P.Y.Wu, *The operator factorization problem*, Lin. Algebra. Appl., **17**(1989), 35–63.

[8] Wang Luqun and Zhang Yongzheng, *On 2-dimensional linear group with ϕ-full morphism rings*, Kexue Tongbao, Vol.28, No.1, 1983.

ϕ- 满射环上 $\mathrm{GL}_n(R)$ 中元素的三角分解

南 基 洙

(哈尔滨工业大学数学系， 150001)

张 永 正

(东北师范大学数学系，长春 130024)

摘 要

本文结果是：设 A 是 ϕ- 满射环 R 上的非拟纯量可逆 $n\times n$ 矩阵，$\beta_j,\gamma_j(1\le j\le n)$ 是 R 中任意元素，它们满足 $\prod_{j=1}^n\beta_j\gamma_j=\det A$, 则存在 n 阶阵 B 和 C 满足 $PAP^{-1}=BC$, 其中 B 是下三角阵， C 是上三角阵，$P\in\mathrm{GL}_n(R)$. 进一步，可以取 B 使 $\beta_j(1\le j\le n)$ 位于 B 的主对角线上，同时可以取 C 使 $\gamma_j(1\le j\le n)$ 位于 C 的主对角线上.