III) $f(t,x,y,z)$ satisfies Nagumo condition which is the same as (III) of Theorem 1 Then BVP(1),(4) has a solution $x(t)$ satisfying

$$|x(t)| \le M, |x'(t)| \le M, |x''(t)| \le N$$

on $0 \le t \le 1$, where N is the same one as Theorem 1.

The proofs of Theorem 2 and Theorem 3 are similar to that of Theorem 1 and are omitted.

Remark 1 The assumption (II) in Theorem 1 can be weakened as follows.

(II)′ there exists a constant $M > 0$, such that

$$xf(t,x,x,0) \ge 0, \quad t \in I,$$

when $|x| > M$.

Remark 2 In order to make the proof convenient and shorten the space of this article, we only consider the homogeneous boundary value conditions (2),(3),(4). But if we make a slight change for the assumptions of the theorems, the same results can also be obtained for nonhomogeneous boundary value conditions.

The authors thank Professor Zhou Qinde for his instruction.

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一类三阶非线性微分方程的边值问题

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摘 要

利用积分 - 微分方程和拓扑度方法讨论了三阶非线性微分方程的若干边值问题，给出了一些简明的解的存在性充分条件.

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A Class of Boundary Value Problems for Third-Oder Nonlinear Differential Equation *

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Abstract Several boundary value problems for third order nonlinear differential equations discussed by means of integral differential equation and topological degree. Some simple sufficient conditions of the existence of solutions are given.

Keywords nonlinear ordinary differential equations, boundary value problems, topological degree.

Classification AMS(1991) 34B15/CCL O175.14

1. Introduction

We consider the third order nonlinear differential equation

$$x''' = f(t, x, x', x'') \tag{1}$$

which is a kind of important equations for fluid mechanics, air dynamics etc. Many results for its boundary value problems have been obtained, for example, [1-3]. But most of them have been done under the assumptions that $f(t, x, x', x'')$ satisfies Nagumo condition and that there exist upper and lower solutions of (1). However, it is difficult to construct upper and lower solutions of (1), meanwhile Nagumo condition given by [1-3] is complicated. In this paper, making use of integral differential equation and topological degree, we study the boundary value problems for the equation (1) with the following three kinds of boundary value conditions

$$x(0) = 0, \begin{cases} a_0 x'(0) - b_0 x''(0) = 0 \\ a_1 x'(1) + b_1 x''(1) = 0 \end{cases} \tag{2}$$

$$x(1) = 0, \begin{cases} a_0 x'(0) - b_0 x''(0) = 0 \\ a_1 x'(1) + b_1 x''(1) = 0 \end{cases} \tag{3}$$

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$$x(c)=0,\left\{\begin{array}{l} a_0x'(0)-b_0x''(0)=0 \\ a_1x'(1)+b_1x''(1)=0 \end{array}\right. \tag{4}$$

where $0<c<1, a_i, b_i \geq 0, a_i+b_i>0 (i=0,1)$. Existence theorems of solutions for BVP(1),(2),(1),(3) and (1),(4) have been given respectively under some simple conditions. Also, the bounds of first order and second order differential functions of their solutions are given. The boundary value conditions (2),(3),(4) include those of [1–3].

2. Main results

Theorem 1 *Assume that*

I) $f(t,x,y,z)\in C(I\times R^3)$, *and* f *is monotonical nonincreasing in* x *for any given* (t,y,z);

II) there exists a constant $M>0$, *such that when* $|x|>M$,

$$xf(t,x,x,0)>0, \quad t\in I;$$

III) $f(t,x,y,z)$ *satisfies Nagumo Condition, that is, there exists a positive value continuous function* $h(s)$ *on* $0\leq s<\infty$, *such that*

$$|f(t,x,y,z)|\leq h(|z|),$$

where $(t,x,y,z)\in I\times[-M,M]\times[-M,M]\times R$, *furthermore*

$$\int_0^\infty \frac{sds}{h(s)}=\infty.$$

Then, BVP(1)(2) has a solution $x(t)$ *satisfying*

$$|x(t)|\leq M, |x'(t)|\leq M, \ |x''(t)|\leq N, \quad t\in I=[0,1],$$

where $N=\max[N_0,2M]$, *and* N_0 *satisfies*

$$\int_M^{N_0}\frac{sds}{h(s)}=2M+1. \tag{5}$$

Proof Let $y(t)=x'(t)$, then BVP(1)(2) is turned to its equivalent boundary value problem of differential-integral equation:

$$y''(t)=f(t,\int_0^t y(s)ds, y(t), y'(t)), \tag*{(1)$'$}$$

$$a_0y(0)-b_0y'(0)=0, a_1y(1)+b_1y'(1)=0. \tag*{(2)$'$}$$

Therefore, we need only to discuss the existence of solution of BVP$(1)'(2)'$. In order to use the method of topological degree, we first give the priori-bounds for BVP$(1)'(2)'$, that is, suppose that BVP$(1)'(2)'$ has a solution $y(t)$, then

$$|y(t)|\leq M, \tag{6}$$

$$|y'(t)|\leq N, \tag{7}$$

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for $t \in [0,1]$. Now we use counterevidince for (6). Suppose, if possible, that $y(t)$ has positive maximum value at a point $t_1 \in I$, and $y(t_1) > M$, then $t_1 \in (0,1)$. Otherwise, if $t_1 = 0$, it is seen that $y'(0) \geq 0$ from the first formula of $(2)'$. However, if $y'(0) > 0$, it conflicts with the assumption that $y(t)$ has maximum at $t_1 = 0$, and on the other hand, if $y'(0) = 0$, then

$$y(0)y''(0) = y(0)f(0,0,y(0),0) \geq y(0)f(0,y(0),y(0),0) > 0.$$

Hence, $y''(0) > 0$, that is, there exists $\delta > 0$, such that $y'(t) > 0, 0 < t < \delta$, this also conflicts with the assumption that $y(0)$ is the positive maximum value. Similarly for $t_1 = 1$, we may obtain a contradiction. Therefore $t_1 \in (0,1)$ and $y'(t_1) = 0$. However, since

$$y(t_1)y''(t_1) = y(t_1)f(t_1, \int_0^{t_1} y(s)ds, y(t_1), 0) \geq y(t_1)f(t_1, y(t_1), y(t_1), 0) > 0,$$

it is seen that $y''(t_1) > 0$, a tradiction. This shows that $y(t) \leq M, t \in I$. Using the same argument, we can obtain $y(t) \geq -M, t \in I$. Thus the inequality (6) follows.

Next we show that inequality (7) holds. Suppose that (7) is not true, that is, there exists $t_1 \in I$, such that $|y'(t_1)| > N$. By Lagrange mean value theorem, there exists $t_0 \in (0,1)$, such that

$$|y'(t_0)| \leq M.$$

Suppose, if possible, that $t_1 > t_0$ (the proof for $t_1 < t_0$ is similar), then there are $\eta, \xi (t_0 \leq \eta < \xi \leq t_1)$, such that

$$|y'(\eta)| = M, |y'(\xi)| = N,$$

and

$$M < |y'(t)| < N, \eta < t < \xi.$$

By assumption (III) and the definition of N, we have

$$\begin{aligned} 2M+1 &\leq \int_M^N \frac{sds}{h(s)} = \int_\eta^\xi \frac{y'(t)y''(t)dt}{h(|y'(t)|)} \leq \int_\eta^\xi |y'(t)dt| = |\int_\eta^\xi y'(t)dt|, \\ &= |y(\xi) - y(\eta)| \leq 2M, \end{aligned}$$

a contradictory result. This prove the inequality (7).

Like the last proof, we consider BVP

$$y''(t) = \lambda f(t, \int_0^t y(s)ds, y(t), y'(t)), \qquad (1)'_\lambda$$

$$a_0 y(0) - b_0 y'(0) = 0, \quad a_1 y(1) + b_1 y'(1) = 0, \qquad (2)'_\lambda$$

where $0 \leq \lambda \leq 1$. Obviously, if BVP$(1)'_\lambda, (2)'_\lambda$ has a solution $y_\lambda(t)$, then $y_\lambda(t)$ satisfies ineqaulity (6),(7). Set

$$\Omega = \{y(t) \in C(I) : |y(t)| < M+1, |y'(t)| < N+1, t \in I\}.$$

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Define a mapping $T_\lambda : \bar{\Omega} \to C^1(I)$

$$T_\lambda(y(t)) = \lambda \int_0^1 G(t,s) f(s, \int_0^s y(\tau)d\tau, y(s), y'(s))ds,$$

here

$$G(t,s) = \begin{cases} \dfrac{1}{a_1 b_0 + a_0 b_1 + a_0 a_1}(a_0 t + b_0)[a_1(s-1) - b_1], & 0 \le t \le s \le 1, \\ \dfrac{1}{a_1 b_0 + a_0 b_1 + a_0 a_1}(a_0 s + b_0)[a_1(t-1) - b_1], & 0 \le s \le t \le 1 \end{cases}$$

is the Green function for $y'' = 0, a_0 y(0) - b_0 y'(0) = 0, a_1 y(1) + b_1 y'(1) = 0$. We easily examine that T_λ is a completely continuous operator and $\theta \notin (id - T_\lambda)(\partial\Omega)$. So we have

$$\deg(id - T_\lambda, \Omega, \theta) = \deg(id - T_1, \Omega, \theta) = \deg(id, \Omega, \theta) = 1.$$

Therefore, BVP(1)$'$(2)$'$ has a solution $y(t)$ satisfying inequalities (6),(7). Furthermore, $x(t) = \int_0^t y(s)ds$ is a solution of BVP(1),(2), and by (6),(7), we also have

$$|x(t)| \le M, |x'(t)| \le M, |x''(t)| \le N, \quad t \in I.$$

The proof is complete.

Theorem 2 *Assume that*

I) $f(t,x,y,z) \in C(I \times R^3)$, *and for any given* (t,y,z), f *is monotonical nondecreasing in* x;

II) there exists a constant $M > O$ *such that*

$$xf(t,-x,x,0) > 0, \quad t \in I,$$

when $|x| > M$;

III) $f(t,x,y,z)$ *satisfies Nagumo condition, that is, the assumption (III) in Theorem 1.*

Then BVP(1),(3) has a solution $x(t)$ *satisfying*

$$|x(t)| \le M, |x'(t)| \le M, |x''(t)| \le N, \quad t \in I,$$

here N *is the same one as Theorem 1.*

Theorem 3 *Assume that*

I) $f(t,x,y,z) \in C(I \times R^3)$ *and is monotonical nondecresing in* x *on* $0 \le t \le c$, *but monotonical nonincreasing in* x *on* $c \le t \le 1$;

II) there exists a constant $M > 0$, *such that*

$$\begin{aligned} xf(t,-x,x,0) &> 0, \quad 0 \le t \le c, \\ xf(t,x,x,0) &> 0, \quad c \le t \le 1, \end{aligned}$$

when $|x| > M$;

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III) $f(t,x,y,z)$ *satisfies Nagumo condition which is the same as (III) of Theorem 1 Then BVP(1),(4) has a solution* $x(t)$ *satisfying*

$$|x(t)| \leq M, |x'(t)| \leq M, |x''(t)| \leq N$$

on $0 \leq t \leq 1$, *where* N *is the same one as Theorem 1.*

The proofs of Theorem 2 and Theorem 3 are similar to that of Theorem 1 and are omitted.

Remark 1 The assumption (II) in Theorem 1 can be weakened as follows.

(II)$'$ there exists a constant $M > 0$, such that

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摘 要

利用积分 - 微分方程和拓扑度方法讨论了三阶非线性微分方程的若干边值问题，给出了一些简明的解的存在性充分条件.

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