Theorem 4 *Extreme disconnectedness and s-closedness are invariant under homeomorphic GOHs.*

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拓扑分子格的极不连通性和 S- 闭性

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摘 要

利用完备余 co-Heyting 代数上的伪补运算，在拓扑分子格上引入极不连通性和 S- 闭性，得到了每个 S- 闭的正则拓扑分子格是极不连通的以及同胚的广义序同态保持极不连通性和 S- 闭性.

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Extreme Disconnectedness and S-closedness of Topological Molecular Lattice *

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Abstract We introduce the concepts of extreme disconnectedness and S-closedness of topological molecular lattices by using the pseudo-negation on complete co- Heyting algebra. Results include: (1) every S-closed regular topological lattice is extremally disconnected; (2) both extreme disconnectedness and S-closedness are invariant under homeomorphic GOHs.

Keywords topological molecular lattice, extreme disconnectedness, S-closedness.

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1. Preliminary

Throughout this paper, $L(M)$(or L) always denotes a completely distributive complete lattice, M the set of molecules. 1 and 0 are the greatest and least elements of $L(M)$ respectively. $\eta \subset L$ is called a co-topology of L, if η satisfies: (1) $0, 1 \in \eta$; (2) η is closed under finite sups and artribary infs. Further $(L(M), \eta)$ is called a topological molecular lattice (briefly: TML). Clearly, η is a complete co-Heyting algebra. Therefore there is the pseudo-negation $*$ in η, as follows.

$\forall P \in \eta$, set $P^* = \wedge\{Q \in \eta : P \vee Q = 1\}$. It has the following properties:

$\forall P, P_i \in \eta, \mathcal{F} \subset \eta$, then

(1) $P^* \vee P = 1$;

(2) if $P_1 \leq P_2$ then $P_2^* \leq P_1^*$;

(3) $P_2^* \leq P_1$ if and only if $P_1^* \leq P_2$; and

(4) $*** = *$;

(5) $(\wedge \mathcal{F})^* = \vee \mathcal{F}^*$.

Proposition 1 *Suppose $(X, \mathcal{U})$ is a topological space, $(\mathcal{P}(X), \mathcal{U}')$ is the TML of $(X, \mathcal{U})$. Then $\forall P \in \mathcal{U}', P^* = P'^-$.*

Remark In general, $P \wedge P^* \neq 0, P \neq P^{**}, (P \vee Q)^* \neq P^* \wedge Q^*$.

Proposition 2 *η is a Boolean algebra if and only if $\forall P \in \eta, P = P^{**}$.*

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$P \in \eta$ is called regular if $P = P^{**}$. Then the notation η_{**} denotes the set of all regular elements of η with the induced order from L.

Proposition 3 *η_{**} is a Boolean algebra.*

Proposition 4 *The following conditions are equivalent:*

(1) η_{**} is a sublattice of η.

(2) $\forall P \in \eta, P^* \wedge P^{**} = 0$.

(3) De Morgan law $(P \vee Q)^* = P^* \wedge Q^*$, is true $\forall P, Q \in \eta$.

(4) $\forall P \in \eta_{**}$, there exists Q such that

$$P \vee Q = 1, P \vee Q = 0.$$

2. Extreme Disconnectedness, S-closedness

Definition 1 *TML$(L(M), \eta)$ is called an extremally disconnected TML, if $\forall P \in \eta$,*

$$P^* \wedge P^{**} = 0.$$

Theorem 1 *Suppose $(L(M), \eta)$ is a topological molecular lattice. Then the following conditions are equivalent.*

(1) $(L(M), \eta)$ is extremally disconnected.

(2) $\forall P \in \eta, P^* \wedge P^{**} = 0$.

(3) $\forall P, Q \in \eta, (P \vee Q)^* = P^* \wedge Q^*$.

(4) η_{**} is a sublattic of η.

(5) $\forall P \in \eta_{**}$, there exists a $Q \in \eta$ such that

$$Q \vee P = 1, \quad P \wedge Q = 0.$$

Proposition 5 *Topological space (X, U) is a extremally disconnected space if and only if the topological molecular lattice $(\mathcal{P}(X), U')$ is extremally disconnected.*

Suppose $(L(M), \eta)$ is a TML. Then η_{**} is closed under finite sups. η_{**} is a base of some co-topology, the co-topology is said to be the semiregularization of η, denoted by η_*. Clearly $\eta_* \subset \eta$.

Proposition 6 *(1) $\forall P \in \eta_*, P^*_{\eta_*} = P^*_{\eta}, P^{**}_{\eta_*} = P^{**}_{\eta}$;*

(2) $\eta_{**} = \eta_{*_{**}}$, that is, η_* has a base consisting of regular elements.

Let $\phi \subset L$. If $\vee\phi = 1$, then we say that ϕ is a cover of L. The meaning of closed cover and regular cover is clear.

Definition 2 *TML$(L(M, \eta))$ is called S-closed if every regular cover has a finite subcover.*

Proposition 7 *Topological space (X, U) is S-closed if and only if its TML$(\mathcal{P}(X), U')$ is S-closed.*

Corolary *Suppose (X, U) is a topological space. Then the followings are equivalent:*

(1) (X, U) is an extremally disconnected space.

(2) every regular open set is a regular closed set.

(3) every regular closed set is a regular open set.

(4) for each closed set $F, F'^{-} \cap F'^{-'-} = \emptyset$.

Proposition 8 *TML$(L(M), \eta)$ is S-closed if and only if its semiregularization $(L(M), \eta_*)$ is also S-closed.*

Dr.Su in [4] introduced the concept of regular TML. The author in [5] gave a characterization of regular TML, as follows:

TML$(L(M), \eta)$ is called regular if and only if $\forall P \in \eta, P = \wedge\{Q \in \eta : P \wedge Q^* = 0\}$.

Theorem 2[5] *TML$(L(M), \eta)$ is regular if and only if $\forall x \in M$ and $F \in \eta$, if $x \not\leq F$, then there exist $P, Q \in \eta$ such that $x \not\leq P$ and $P \wedge Q = 1, F \wedge Q = 0$.*

Theorem 3 *Every both S-closed and regular TML is extremally disconnected.*

Proof Taking $P \in \eta$, then $P = \wedge\{Q \in \eta : P \wedge Q^* = 0\}$. Thus

$$P^* = \vee\{Q^* \in \eta : P \wedge Q^* = 0\},$$

and

$$\{P^{**}\} \cap \{Q^* \in \eta : P \wedge Q^* = 0\}$$

is a regular cover of L. So there exist $Q_1^*, \cdots, Q_n^*$ such that

$$P^{**} \vee Q_1^* \vee \cdots \vee Q_n^* = 1,$$

where $P \wedge Q_i^* = 0, i = 1, \cdots, n$. Since $P^{**} = P^{***} \leq Q_1^* \vee \cdots \vee Q_n^*$, then

$$P^* \wedge P^{**} \leq P^* \wedge P \leq (Q_1^* \vee \cdots \vee Q_n^*) \wedge P.$$

So $P^* \wedge P^{**} = 0, (L(M), \eta)$ is extremally disconnected.

3. Generalized Order-Homomorphism

GOH(Generalized Order-Homomorphism) $f : (L_1(M_1), \eta_1) \to (L_2(M_2), \eta_2)$ is called to be continuous, if $\forall P \in \eta_2, f^{-1}(P) \in \eta_1$. Bijective GOH f is called a homeomorphism if and only if f and f^{-1} are continuous. GOH f is called closed if for each closed element $P, f(P)$ is closed.

Lemma 1 *Bijective GOH is a homeomorphism if and only if it is both continuous and closed GOH.*

Lemma 2 *Suppose $f : (L_1(M_1), \eta_1) \to (L_2(M_2), \eta_2)$ is a hemeomorphisc GOH, then*

(1) $\forall P \in \eta_2, f^{-1}(P^*) = (f^{-1}(P))^*$;

(2) $\forall A \in \eta_1, f(A)^* = f(A^*)$;

(3) $A \in \eta_1, A$ is regular if and only if $f(A)$ is regular;

(4) $P \in \eta_2, P$ is regular if and only if $f^{-1}(P)$ is regular.

Theorem 4 *Extreme disconnectedness and s-closedness are invariant under homeomorphic GOHs.*

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利用完备余 co-Heyting 代数上的伪补运算，在拓扑分子格上引入极不连通性和 S- 闭性，得到了每个 S- 闭的正则拓扑分子格是极不连通的以及同胚的广义序同态保持极不连通性和 S- 闭性.