367- 371.

[2]N. Jacobson, *Basic Algebra II, Chapter 2: Universal Algebra*, W. H. Freeman and company San Francisco, 1980.

[3]Yu Yongxi, *The quasikernels in an n- preadditive category*, J. Math. Res. Exposition, First Issue(1981), 7- 15.

[4]Yu Yongxi, *Union and Intersection of a quasikernel family*, Kexue Tongbao, **28**: **11** (1983), 1437- 1440.

[5]B. Mitchell, *Theory of Categories*, New York, 1965.

[6]M. Barr, *Exact categories*, Lecture Notes in Math., Springer, Berlin, **236**(1971), 1 - 120.

[7]R. Goldblatt, Topoi, *The Categorial Analysis of Logic*, North- Holland Publishing Company. Amsterdam. New York. Oxford, 1984.

[8]J. Michalski, *On some functors from the category of n- groups*, Bull. Acad. Pol. Sci Vol. XXVII, No. 6(1979), 437- 441.

[9]Yu Yongxi, *The category of left R- n modules and Hom functors*, J. Math. Res. Exposition, **4**(1982), 21- 30.

[10]P. A. Grilet, *Regular categories*, Lecture Notes in Math., Springer, Berlin, **236**(1971), 121- 222.

[11]Wang Shaowu, Yu Yongxi, *Some Conclusions on Categories with Terminal Objects*, *J. Math. Res. Exposition*, 4(1996), 497- 504.

# 在具有终对象的范畴内的正合列

**王 少 武　　　　于 永 溪**

(**锦州师范学院管理学院**, 121003) (**苏州大学数学科学学院**, 215006)

## 摘　要

**本文对于具有终对象的范畴定义了四个一般正合列. 在*Abel* 范畴同调代数里, 对于具有零对象的任何范畴四个一般正合列与通常的正合列相同.**

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# Exact Sequences in Categories with Terminal Objects*

*Wang Shaowu* *Yu Yongxi*
(Jinzhou Teachers' College, 121003) (Suzhou University, 215006)

**Abstract** In this paper, we define four generalized exact sequences for the categories with terminal objects. For any category with a null object, the above four exact sequences coincide with the ordinary exact sequences in HAA (homological algebra in abelian categories).

**Keywords** image, pullback, exact sequence, equivalency.

**Classification** AMS(1991) 18B99/CCL O154

The category $C - c_r$, which is defined in [1] for describing a changing process of a system, possesses a terminal object and has finite products. Some of its subcategories have some interesting properties, for example, one of them satisfies Axiom (P) (see [1]). It is the task of C. T. S discussing the category $C - c^r$. One problem is: can we establish its homological theory or homological algebra of some of its subcategories as a tool for the classification of its morphisms?

Since a morphism $f$ in $C - c^r$ is a process from one state of affairs of a system to another: $c_1 \xrightarrow{f} c_2$, we hope that our exact sequences are of the form $c_1 \xrightarrow{g} {}_2 \xrightarrow{f} c_3$.

There are many categories with terminal objects which are related to abelian category in varying degrees, and that the additive operation in an abelian category creates a null object, can we then find, within the scope of HAA, anything substantially independent of null objects and extend it to some categories with terminal objects? Where HAA means homological algebra in abelian categories.

Thus, the above mentioned origins give rise to the following problem:

To what degree and to which categories with terminal objects can HAA be naturally extended? By "naturally" we mean that the generalized theory degenerates into some facts of HAA when the terminal objects are null.

This problem is called the natural homological algebra problem or NHA for short. To answer NHA, we defined the quasikernels for the categories with terminal objects as follows (see [3]):

Let $F$ be a terminal object, if the diagram $\begin{tikzcd} \bullet \arrow[r, "s"] \arrow[d, "u"'] & F \arrow[d, "h"] \\ \bullet \arrow[r, "f"'] & \bullet \end{tikzcd}$ is a pullback, then the morphism $u$ is called a (terminal) quasikernel of $f$ and $h$ is called a lateral of $u$. We write $QK^F(f)$ for the class of all quasikernels of $f$.

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A terminal object $F$ is called to be quasinull, if for any object $A$ the morphism $t: A \to F$ is epi

In this paper, the symbols are the same as those in [11].

**Axiom (P)** If there is a commutative diagram $\begin{tikzcd} \bullet \arrow[r, "v"] \arrow[d, "u"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "f"'] & \bullet \end{tikzcd}$, with $g$ a monic, then there is a pullback $\begin{tikzcd} \bullet \arrow[r, "b"] \arrow[d, "a"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "f"'] & \bullet \end{tikzcd}$

In Set (the category of sets) and many categories based upon sets, the following axiom is a basic fact:

**Axiom (U)** If $f: A \to B$ is a regular epi and $a_i$ is a subobject of $\$B$, $i \in T$, such that

$$\widetilde{\bigcup_T} a_i = (1),$$

then

$$\widetilde{\bigcup_T} f^{-1}(a_i) = (1).$$

The sense of $\widetilde{\cup}$ has been shown in detail in [11].

A category is called a (P)- category, if for it Axiom (P) holds (see [11] in detail).

A regular epi is called to be (U)- epi, if for it Axiom (U) holds; A category is called a (U)- category, if every regular epi in it is (U)- epi; For a quasiregular (see [11], §1]) category, a morphism $f$ is called a (U)- morphism, if the regular epi $e$, which is in the regular factorization $f = me$, is (U)- epi

In this paper, we will define four generalized exact sequences corrsponding to the quasikernels by imitating the idea of HAA. They are natural. The main theorem is Theorem 2, for one thing, as far as (P) (U)- quasiregular categories it makes the structure of the generalized exact sequences clear, an exact sequence at $A$ is a family of exact sequences by components at $A$ and a normal exact sequence in HAA is a family which consists of only one exact sequence by a component. For another, since an exact sequence by a component approximates to a normal exact sequence in HAA, and at this moment the terminal object $F$ approximates to a null object, we can imitate HAA to establish some homologous theoris, therefore the theorem provides a new premise for our further discussion. Thus, in this paper we can have a general view of crux of NHA of the (P) (U)- quasiregular categories

Now we define the generalized exactness and show the equivalence and the natruality of them. For now on we appoint that the categories always have terminal objects

**Definition 1** *A sequence* $A \xrightarrow{g} B \xrightarrow{f} C$ *is called to be I- exact at* $B$, *if* $g$ *has an image and there is a family* $\{a_i\}_{i \in T}$ *of subobjects* $a_i$*'s of* $A$ *such that* $\cup_T a_i = \langle 1 \rangle$ *and* $\{g(a_i)\}_T = QK^F(f)$ (*see* [11]).

**Definition 2** *A sequence* $A \xrightarrow{g} B \xrightarrow{f} C$ *is called to be I'- exact at* $B$, *if* $g$ *has an image and for each* $\langle u_i \rangle \in QK^F(f)$ *exists the inverse image* $g^{-1}(u_i) = \langle a_i \rangle$ *such that* $\cup_j a_i = \langle 1_A \rangle$ *and*

$\{g(a_i)\}_i = QK_F(f)$.

**Definition 3** *A sequence* $A \xrightarrow{g} B \xrightarrow{f} C$ *is called to be II- exact at* $B$, *if* $\widetilde{\bigcup}_{u_i \in QK^F(f)} u_i$ *exists and* $\widetilde{\bigcup}_{u_i \in QK^F(f)} u_i = \mathrm{Im}(g)$ (*see* [11]).

**Definition 4** *Let* $\langle u \rangle \in QK^F(f)$, *a sequence* $A \xrightarrow{g} B \xrightarrow{f} C$ *is called to be exact by the component* $\langle u \rangle$ *at* $B$, *if* $\mathrm{Im}(g) = \langle u \rangle$.

Each of the four exact sequences is called to be generalized exact.

From now on, the alphabets in the above four definitions will get the same senses as above, respectively.

**Proposition 1** *For any category with a null object, the above four exact sequences coincide with the ordinary exact sequences in HAA.*

**Proof** Given a sequence which is I- exact at $B$, we hope to prove that it is ordinary exact. Since $\{g(a_i)\} = QK^0(f) = \{\langle u \rangle\}$, we have $g(a_i) = \langle u \rangle$. By the definition of I-exactness, $\mathrm{Im}(g)$ exists, and hence $\mathrm{Im}(g) = g(1) = g(\cup a_i)$. Thus, Proposition 2.2 (see [11]) says that $\mathrm{Im}(g) = g(\cup a_i) = \widetilde{\cup} g(a_i) = \widetilde{\cup} u = \langle u \rangle$, that is, the sequence is ordinary exact.

Given an ordinary exact sequence, we are going to prove that it is I'- exact. In fact, we have $\mathrm{Im}(g) = \ker(f) = \langle u \rangle$, and because $\mathrm{Im}(g) = \langle u \rangle$, there is a morphism $s$ such that $g = us$, since $u$ is monic, we have a pullback $(g, u; 1, s)$ and so that $g^{-1}(u) = \langle 1 \rangle$. Let $a = g^{-1}(u)$, then $\{g(a)\} = \{Im(g)\} = \{\langle u \rangle\} = QK^0(f)$, and hence the sequence is I'- exact. Since an I'- exact sequence must be I- exact, we have proved that the I'-exactness and the I- exactness coincide with the ordinary exactness in HAA.

It is clear that the II- exactness and the exactness by a component coincide with the ordinary exactness. □

[4, Theorem 2] showed that for a category with a quasinull terminal object an II-exact sequence is not exact by a component when $QK^F(f)$ *consists of more than one element.*

**Proposition 2** *For a category with a quasinull terminal object, if the sequence* $A \xrightarrow{g} B \xrightarrow{f} C$ *is I'- exact at* $B$, *then the following are equivalent:*

1. $\cap a_i$ *exists;* 2. $\cap u_i$ *exists;* 3. $QK^F(f)$ *consists of only one element.*

The proof can easily be completed by [4, Theorem 1], but it needs to be noticed that the condition "has image" in [4, Theorem 1] may be omitted.

This proposition shows that though there is no requirement on the existence of $\cap a_i$ *in the above definitions of generalized exactnesses, the intuitive fact, whether* $\cap a_i$ *exists or not in* $AG_{r_n}$ *and* $_RM_n^1$ (*see* [11]), *is implied in the definitions.*

**Proposition 3** *For a* $(P)$- *category, the I- exactness is equivalent to the I'- exactness.*

**Proof** Suppose the sequence is I- exact, then, for each $a_i$, *since* $g(a_i) = \langle u \rangle_i$, *Axiom* $(P)$ *shows* $g^{-1}(u_i)$ *exists. Let* $g^{-1}(u_i) = \langle b_i \rangle$, *then* $b_i \geq a_i$. *Thus, if* $b_i$ *is carried into* $h$ *by* 1, *then* $a_i$ *is carried into* $h$ *by 1 also, and since* $\cup a_i = \langle 1 \rangle$, *Lemma* 1.4 (*see* [11]) *shows* $h$ *is an isomorphism, so that to use Lemma* 1.4 *again we have* $\widetilde{\cup} b_i = \langle 1 \rangle$, *further, by Lemma* 1.3 (*see* [11]) *we have* $\cup b_i = \langle 1 \rangle$

We are going to prove that $g(b_i)$ exists and $g(b_i) = \langle u_i \rangle$. Since $g(a_i) = \langle u_i \rangle$, $b_i \geq a_i$, and $\langle b_i \rangle = g^{-1}(u_i)$, we can say $ga_i = u_i s_i$, $a_i = b_i t_i$, and $gb_i = u_i m_i$, respectively. If there are two morphisms $a$ and a monic $b$ such that $gb_i = ba$, then we have $ga_i = g.(b_i t_i) = (ba) t_i = b.(a t_i)$, so since $g(a_i) = \langle u_i \rangle$, the definition of images shows there is a morphism $n_i$ such that $bn_i = u_i$, so that $g(b_i) = \langle u_i \rangle$, and hence we have proved the sequence is I'-exact. □

**Proposition 4** *For any category, an I-exact sequence must be II-exact.*

**Proof** In fact, this is a corollary of proposition 2.2 (see [11]).

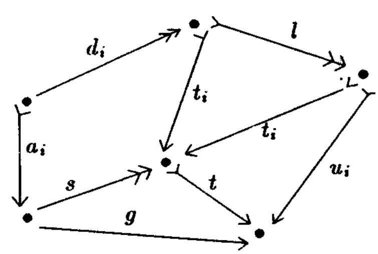
**Theorem 1** *For any (P)-category, if $g$ is monic, then I-exactness, I'-exactness, and II-exactness are equivalent.*

**Proof** Since an I'-exact sequence must be I-exact and we have Proposition 4, it is enough for the proof to demonstrate an II-exact sequence is I'-exact.

Since $g$ is monic and the sequence is II-exact, we have $\langle g \rangle = \mathrm{Im}(g) = \tilde{\cup} u_i$, hence for each $u_i \in QK^F(f)$ there is a unique $a_i$ such that $ga_i = u_i$, and so that $g(a_i) = u_i$. Easy to check the diagram $(g, u_i; a_i, 1)$ is a pullback, we know $\langle a_i \rangle = g^{-1}(u_i)$. Lemma 1.5 (see [11]) shows $\cup a_i = \langle 1 \rangle$ and Lemma 1.3 (see [11]) shows $\cup a_i = \langle 1 \rangle$. Therefore, the sequence is I'-exact and the proof is complete. □

**Theorem 2** *For any (P)-quasiregular category, if $g$ is a (U)-morphism, then I'-, I-, and II-exactness are equivalent.*

**Proof** Given an II-exact sequence, we discuss the diagram




Let $g = ts$ be a regular factorization of $g$, then we have $(t) = \mathrm{Im}(g)$. Since $\mathrm{im}(g) = \tilde{\cup} u_i$, for each $i$ there is a morphism $t_i$ such that $tt_i = u_i$, so that the diagram $(t, u_i; t_i, 1)$ is a pullback. By Axiom (QR) (see [11]), there is a pullback $(s, t_i; a_i, d_i)$ with $d_i$ a regular epi, so that $a_i = s_{-1}(t_i)$. Now we have a pullback $(g, u_i; a_i, 1d_i)$ and $\langle a_i \rangle = g^{-1}(u_i)$. Lemma 1.5 (see [11]) tells us $\tilde{\cup} t_i = \langle 1 \rangle$, and Axiom (U) shows $\tilde{\cup} s^{-1}(t_i) = \langle 1 \rangle$, that is, $\tilde{\cup} a_i = \langle 1 \rangle$. By Lemma 1.3 (see [11]) we know $\cup a_i = \langle 1 \rangle$. Moreover, $ga_i = u_i(1d_i)$ is a regular factorization of $ga_i$, so that $g(a_i) = \langle u_i \rangle$. Therefore, the sequence is I'-exact. □

## References


[1] Yu Yongxi, *Some properties on the category $C$-$c^r$*, J. Math. Res Rxposition, **3** (1987),