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半非紧测度与集值AM 映象

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摘　要

本文以半非紧测度为工具研究一类非线性集值映象的性质，然后把所得结果用于证明微分包含的解的存在性

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Measures of Semi-Noncompactness and Set-Valued AM-Mappings*

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Abstract In this paper we define measures of semi-noncompactness in a locally convex topological linear space with respect to a given seminorm. Then we get a fixed point theorem for a class of condensing set-valued mappings and apply it to differential inclusions.

Keywords ordered topological linear space, almost order-bounded set differential inclusion.

Classification AMS(1991) 46A, 47H/CCL O177.1

Recently, many papers are concerned with existences of fixed points for set-valued contraction mappings, see, e.g., [9] and reference therein. In this paper, motivatied by the ideas of [7], we introduce a notion of measure of semi-noncompactness for subsets of a topological vector lattice which not only reduces to that given in [7] for the norm topology on a Banach lattice, but which applies equally to the weak topology in a wide class of Banach lattices. These ideas lead naturally to considering a class of set-valued mappings which we call set-valued AM-mappings and for which we prove a fixed point theorem (Theorem 3).

Let (E, τ) be a real and locally convex topological linear space. We assume that there is an order relation $\leq$ in E, which makes E a vector lattiec. For $x \in E$. Let

$$x^+ = x \vee 0,\ x^- = (-x) \vee 0,\ |x| = x^+ + x^-,\ E_+ = \{x \in E \mid x \geq 0\}.$$

We assume that the topology τ and the partial order $\leq$ satisfy the following condition (H):

(H) Let $x \in E$. If $\{x_n\} \subset E$ is a sequence which is τ-convergent to x, then there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ and there exist elements $y, z \in E$ with $0 \leq x^+_{n_j} \leq y$, $0 \leq x^-_{n_j} \leq z$ such that the subsequence $\{x^+_{n_j}\}$ (respectively $\{x^-_{n_j}\}$) is τ-convergent to y (respectively z).

It is clear that condition (H) implies that the positive cone E_+ is τ-closed in E. We remark that if E is a Banach lattice and if τ is the norm topology on E, then condition (H) is clearly satisfied since the lattice operations are continuous for the norm topology. If τ is the weak topology on a Banach lattice then the situation is somewhat different, since that lattice operations are, in general, not weakly continuous. However, condition (H) will be satisfied for the weak topology of a

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Banach lattice E if the solid hull of any weakly compact subset of E is again relatively weakly compact, in particular, if E is reflexive or if E is an abstract L-space.

Let φ be a seminorm in E, which is lower semicontinuous with respect to τ, that is, if $B_\varphi = \{x \in E \mid \varphi(x) \leq 1\}$, then B_φ is τ-closed. In addition, we suppose that φ is monotone with respect to the given partial order, that is, $0 \leq x \leq y$ implies $\varphi(x) \leq \varphi(y)$. If $D \subset E$, and if there exists $r > 0$ such that $D \subset rB_\varphi$, then D is called φ-bounded, D is called almost order-bounded relative to φ, if for given $\varepsilon > 0$, there exists $u \in E_+$ such that $D \subset [-u, u] + \varepsilon B_\varphi$. This is equivalent to the statement; for given $\varepsilon > 0$: there exists $u \in E_+$ such that

$$\varphi((|x| - u)^+) \leq \varepsilon, \forall x \in D.$$

We remark that if E is an abstract L-space and if φ is the given norm on E then a subset $D \subset E$ is almost order-bounded relative to φ if and only if D is relatively weakly compact.

For any φ-bounded subset D in E, define

$$\rho_\varphi(D) = \inf\{\delta > 0 \mid \exists u \in E_+ \text{ such that } D \subset [-u, u] + \delta B_\varphi\}.$$

It is easily seen that [5]

$$\rho_\varphi(D) = \inf\{\delta > 0 \mid \exists u \in E_+ \text{ such that } \varphi((|x| - u)^+) \leq \delta, \forall x \in D\}.$$

We say that $\rho_\varphi(D)$ is the measure of semi-noncompactness of D with respect to φ. We will omit φ if there is no danger of confusion. Our definition is motivated by the measure of semi-noncompactness introduced by de Pagter and Schep[7] and reduces to theirs for the case that E is a Banach lattice with φ the given norm on E and τ the norm topology.

We now gather some simple properties.

Lemma 1 *If D, D_1, D_2 are φ-bounded sets in E, then*

(i) $\rho(D) = 0 \Leftrightarrow D$ *is almost order-bounded* (*relative to* φ);

(ii) $\rho(D_1 + D_2) \leq \rho(D_1) + \rho(D_2)$, $\rho(\lambda D) = |\lambda| \rho(D)$, λ real;

(iii) $D_1 \subset D_2 \Rightarrow \rho(D_1) \leq \rho(D_2)$;

(iv) $\rho(D \cup \{x_0\}) = \rho(D)$, $x_0 \in E$;

(v) $\rho(\overline{D}) - \rho(D)$, *where $\overline{D}$ denotes the closure of D with respect to τ*;

(vi) $\rho(\overline{co}(D)) = \rho(D)$, *where $\overline{co}(D)$ is τ-convex closure of D.*

Proof (i), (iii) and (iv) are clear.

(ii) Let $x_k \in D_k$, $k = 1, 2$. Given $\varepsilon > 0$, there exists $a_k \geq 0$ such that

$$D_k \subset D[-u_k, u_k] + (\rho(D_k) + \frac{\varepsilon}{2})B_\varphi, k = 1, 2.$$

If $x_k = y_k + z_k$, with $y_k \in [-u_k, u_k]$ and $z_k \in (\rho(D_k) + \frac{\varepsilon}{2})B_\varphi$, $k = 1, 2$, then

$$x_1 + x_2 \in [-(u_1 \vee u_2), u_1 \vee u_2] + (\rho(D_1) + \rho(D_2) + \varepsilon)B_\varepsilon.$$

It follows that

$$\rho(D_1 + D_2) \leq \rho(D_1) + \rho(D_2) + \varepsilon,$$

for every $\varepsilon > 0$ and (ii) follows.

(v) From (iii), it follows that $\rho(D) \leq \rho(\overline{D})$. To prove the reverse inequality if $\varepsilon > 0$ is given, then there exists $u \in E_+$ such that

$$\varphi((|x|-u)^{+})\leq\rho(D)+\varepsilon,\ \forall x\in D.$$

If $x\in\overline{D}$, there exists a sequence $\{x_n\}\subset D$, with $x_n\to x$ in τ. By (H), there exists a subsequence $\{x_{n_j}\}\subset\{x_n\}$ satisfying: $x_{n_j}^{+}\to y$, $x_{n_j}^{-}\to z$, and $x^{+}\leq y$, $x^{-}\leq z$. So $|x_{n_j}|-u\to v-u$ in τ, where $v=y+z$. By (H) again, there eixsts a subsequence $\{x_{n_j^1}\}$ of $\{x_{n_j}\}$ such that

$$(|x|-u)^{+}\leq(v-u)^{+}\leq\lim_{j'}(|x_{n_{j'}}|-u)^{+}.$$

From lower semicontinuity of φ relative to τ and the fact that φ is monotone, we have

$$\varphi((|x|-u)^{+})\leq\varphi((v-u)^{+})\leq\varphi(\lim_{j'}(|x_{n_{j'}}|-u)^{+})\leq\liminf_{j'}\varphi(|x_{n_{j'}}|-u)^{+})$$
$$\leq\rho(D)+\varepsilon,\ \forall x\in\overline{D},$$

and the conclusion follows

(vi) By (iii) and (v), it is only necessary to prove $\rho(\mathrm{co}(D))\leq\rho(D)$. For any $\alpha>\rho(D)$, there exists $u\in E_{+}$ such that

$$D\subset[-u,u]+\alpha B_{\varphi}$$

Because $[-u,u]+\alpha B_{\varphi}$ is a convex set, it follows that

$$\mathrm{co}(D)\subset[-u,u]+\alpha B_{\varphi}$$

So $\rho(co(D))\leq\alpha$. Let $\alpha|\rho(D)$ and we get the conclusion.

We now give the following:

Definition 2 *Let (E,r) be a locally convex real linear topological space, which is in addition a vector lattice such that condition (H) is satisfied, and let φ be a monotone seminorm in E. We say that a set-valued mapping F is upper semi-continuous at $x_0\in E$, if for any neighbourhood $N(F(x_0))$ of $F(x_0)$, there exists a neighbourhood $N(x_0)$ of x_0 such that*

$$\forall x\in N(x_0),\ F(x)\subset N(F(x_0)).$$

F is called upper semi-continuous if F is upper semi-continuous at every point $x\in E$. Let D be a subset of E and $F:D\to E$ be a τ-upper-continuous set-valued mapping, which maps φ-bounded sets to φ-bounded set. If F maps each φ-almost order bounded subset of D to a τ-relatively compact set, then F is called a set-valued AM-mapping on D. If for any φ-bounded set $S\subset D$, the condition $\rho(S)>0$ implies $\rho(F(S))<\rho(S)$, then F is called a condensing set-valued AM-mapping.

Lemma 3[4] *Let S be a compact Hausdorff space and $F:S\to S$ a closed set-valued mapping (that is, its graph is a closed subset of $S\times S$), then F is upper-semicontinuous.*

Theorem 4 *Suppose D is a non-empty, φ-bounded and τ-closed convex subset in E, if $F:D\to D$ is a condensing set-valued AM-mapping, then F has a fixed point in D.*

Proof *Let $x_0\in D$. Let Z be the collection of all τ-closed convex subsets of D containing x_0 and being invariant under F. Because $D\in Z$, Z is non-empty. If $S_0=\bigcap_{S\in Z}S$ then $x_0\in S_0\subset D$, S_0 is τ-closed and convex, $F(S_0)\subset S_0$, and $co\{F(S_0),x_0\}\subset S_0$ and consequently*

$$F(\overline{\mathrm{co}}\{F(S_0),x_0\})\subset F(S_0)\subset\overline{\mathrm{co}}\{F(S_0),x_0\}.\tag{1}$$

By (1), $\overline{co}\{F(S_0),x_0\}\in Z$, and from the definition of S_0, we have

$$\overline{\mathrm{co}}\{F(S_0),x_0\}=S_0.\tag{2}$$

It follows from Lemma 1 that

$$\rho(S_0) = \rho(\overline{\text{co}}\{F(S_0), x_0\}) = \rho(\{F(S_0), x_0\}) = \rho(F(S_0)).$$

As F is condensing, $\rho(S_0) = 0$, so that S_0 is an almost order-bounded set, and consequently $F(S_0)$ is τ-relatively compact since F is a set-valued AM-mapping. From (2), S_0 itself is τ-compact. By the Kakutani-Fan fixed-point theorem [4], F has at least one fixed point in $S_0 \subset D$ and the proof is complete.

We now give an example for application of Theorem 4.

Let $CC(X)$ denote the class of closed convex subsets of Banach space X. Consider the differential inclusion in $Y = L^1(0, 1)$:

$$x(t) \in f(t, x(t)), \ t \in (0, 1). \tag{3}$$

$$x(0) = x_0. \tag{4}$$

Suppose

(a1) $f(t, z) \in CC(R)$ for every $t \in (0, 1)$ and $z \in R$;

(a2) $t \to f(t, z)$ is a measurable set-valued function for $z \in R$;

(a3) There exist a nonnegative function $a(t) \in Y$ and a constant $b \geq 0$ such that

$$|y(t)| \leq a(t) + b|x(t)|, \text{ for every } y \in M_u(f_x),$$

where $M_u(f_x)$ is the set of all measurable selections of $f(t, x(t))$, $t \in (0, 1)$, $x \in L^1(0, 1)$;

(a4) For any closed convex subset S of Y. $\{(x, f(\cdot, x)) | x \in S \in CC(Y \times Y)$.

Lemma 5 Suppose (a1), (a2) hold, then

$$H(g) = h \in Y | h(t) \in F_g(t) \equiv f(t, x_0 + \int g(s) ds), \text{a. e. } t \in (0, 1) \in CC(Y)$$

for every $g \in Y$.

Proof The convexity of $H(g)$ is easy to prove.

Next we show that $H(g)$ is closed. What we do here is somewhat more than we need. In fact, we prove $H(g)$ is weakly sequencially closed.

Let $h_n \in H(g)$, $n = 1, 2, \cdots$, $h_n \to h_0$ weakly in Y. Since $h_n \in H(x)$, $h_n(t) \in F_g(t)$, a. e., coonsequently, by the definition of integral for set-valued functions and $F_g(t) \in CC(R)$, $t \in (0, 1)$, we have

$$\frac{1}{m(J)} \int F_g(t) dt$$

is a convex set, and $\frac{1}{m(J)} \int_J h_n(t) dt \in \frac{1}{m(J)} \int_J F_g(t) dt$ for all n, where $m(\cdot)$ is Lebesque measure and $J \subset (0, 1)$ is any measurable subset with $m(J) > 0$.

By Mazur theorem,

$$\lim \frac{1}{m(J)} \int h_n(t) dt = \frac{1}{m(J)} \int h_0(t) dt \in \frac{1}{m(J)} cl(\int F_g(t) dt)$$

hold for every measurable $J \subset (0, 1)$, where by cl$(\cdot)$ we mean the closure in norm topology. Therefore $h_0(t) \in \overline{co} f(t, x(t)) = F_g(t)$, a. e., It follows that $h_0 \in H(g)$. So $H(g) \in CC(Y)$, $g \in Y$.

Theorem 6 bxSuppose (a1)-(a4) hold and $0 \leq b < 1$, then problem (3)-(4) has a solution.

Proof For every $g \in Y$, $H(g) \in CC(Y)$. Let

$$B_r = \{y \in Y \mid \|y\| \leq r\},$$

where $r = \|a\|(1-b)^{-1}$. Then $H: B_r \to B_r$. Let us prove that $H: B_r \to B_r$ is condensing. Suppose $Q \subset B_r$ and $\rho(Q) > 0$. For any $\varepsilon > 0$, there exists a nonnegative $u \in Y$ such that

$$Q \subset [-u, u] + [\rho(Q) + \varepsilon] B_1.$$

So for every $x \in Q$, it has a decomposition:

$$x = y + z, y \in [-u, u], z \in [\rho(Q) + \varepsilon] B_1.$$

So

$$H(x) \subset [-a - bu - b|x_0|, a + bu + b|x_0|] + b[\rho(Q) + \varepsilon] B_1,$$

Where $u_1(t) = \int u(s) ds$. It follows that

$$\rho(H(Q)) \leq b(\rho(Q) + \varepsilon).$$

Letting $\varepsilon \to 0$ we get the conclusion that H is condensing. By the proof of Theorem 4 we can get $S_0 \in CC(Y)$ such that $H: S_0 \to S_0$ and S_0 is an almost order-bounded set. Since the relatively weakly compact subset of $L^1(0, 1)$ are precisely those which are almost order-bounded for norm topology ([5]), S_0 is weakly compact. By the assumption (a4) the graph of $H|S_0$ is weakly closed in $Y \times Y$, therefore $H: S_0 \to S_0$ is upper-semicontinuous in the weak topology in $L^1(0, 1)$, here $H|S_0$ is the restriction of H to S_0. By Theorem 4, H has a fixed point y in S_0. Obviously, $y(t) \in f(t, x_0 + \int y(s) ds)$, $t \in (0, 1)$, a. e. Define

$$x(t) = x_0 + \int y(s) ds, \quad t \in (0, 1),$$

then $x(0) = x_0$, $x'(t) = y(t) \in f(t, x(t))$, a. e., $t \in (0, 1)$. The proof is complete.

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