${}^2F_4(q)$ is also $7'$-group by lemma 1, therefore, ${}^2F_4(q)$ can not be inner 7-closed group, and (2.8) holds.

(2.9) G is not of type ${}^2G_2(q)$, $q = 3^{2m+1}$, $m \geq 1$, ${}^2D_n(q)$, $n > 3$.

Let $X = {}^2G_2(q)$, $S \in Syl_2(X)$, by [12, p. 292], one has

$$C_X(S) = S, \ |N_X(S)| = 168,$$

it follows that $N_X(S)$ is not 7-closed group and ${}^2G_2(q)$ can not be inner 7-closed group. For ${}^2D_n(q)$, with the same argument before, we need to consider only ${}^2D_4(q)$. It is known that $A_2(q)$ and $A_1(q) * {}^2D_3(q)$ are Levi subgroups of ${}^2D_4(q)$. For a same q, both $A_2(q)$ and ${}^2D_3(q)$ are not $7'$-group at the same time by lemma 1, hence ${}^2D_4(q)$ can not be 7-closed group, and (2.9) holds.

The proof of the Theorem is complete by the classification theorem of finite groups.

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关于内 7-闭单群的结构

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摘　要

研究内 p-闭群的结构是一个很活跃的课题. 对于 $p = 2, 3, 5$ 的内 p-闭群的结构已经被确定(见 [1, 2, 3]). 本文确定内 7-闭单群的结构.

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On the Structure of Inner 7-Closed Simple Groups*

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Abstract The structure of inner p-closed groups for $p = 2, 3, 5$ are known (see [1, 2, 3]). In this paper, we shall dermine the structure of the inner 7-closed simple groups

Keywords p-closed group, simple group.

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1. Introduction

Throughout this paper, G denotes a finite group and p is a prime. G is said to be p-closed if the p-subgroup of G is normal in G. G is called an inner p-closed group if every proper subgroup of G is Sp-closed and G itself is not p-closed. For the structure of inner p-closed groups, Chen Zhongmu[1] showed that an inner p-closed group G must be one of the two types: (1) $G/\Phi(G)$ is non-abelian simple group, where $\Phi(G)$ denotes the Frattini subgroup of G; (2) G is a q-basic group of order $p^{\alpha}q^{\beta}$ for some prime q. [1] also determined that the inner 2-closed groups are dihedral. For the inner p-closed simple groups G, Li Shirong[2] has shown that G is isomorphic to PSL $(2, 2^r)$, r is odd prime, for $p = 3$; You Taijie[3] has proved that G is isomorphic to either PSL $(2, 5)$ or $Sz(2^r)$, r is odd prime, for $p = 5$; Recently, Xiao Wenjun[4] obtained an important result that inner p-closed simple group has cyclic Sp-subgroup for odd p, and he pointed out that it is difficult to determine the structure of inner p-closed simple groups for $p \geq 5$. The aim of this paper is to prove the following.

Theorem *An inner 7-closed simple group is isomorphic to one of the following groups:* $A_1(7), A_1(p)$,
$p \equiv -1 \pmod 7$, $A_1(p^3)$ $p \equiv 3$ *or* $5 \pmod 7$.

2 The Proof of the Theorem

Lemma 1 *Let G be a non-abelian simple 7'-group, then G is isomorphic to one of the following groups:*

(1) *The sporadic groups M_{11}, M_{12}, J_3, and the alternating groups A_5, A_6.*

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(2) $A_1(q), B_2(q)$, $q = p^f$, $(3, f) = 1$, $q \not\equiv 0, 1$ and -1 (mod 7);

(3) $A_n(q)$, $n = 2, 3, 4$, $q \equiv 3, 5$ (mod7), $q = p^f$, $(6, f) = 1$;

(4) ${}^2A_n(q)$, $n = 2, 3, 4$, $q = p^f$, $q \equiv 2, 4$(mod7), $(6, f) = 1$;

(5) ${}^2F_4(2^{2m+1})$, ${}^2B_2(2^{2m+1})$, $m \not\equiv 1$ (mod3), where ${}^2F_4(2)'$ *instead of* ${}^2F_4(2)$. **Proof** We will use the classification theorem of finite simple groups. By directly checking the order of the simple groups, it is seen that there are only five groups $M_{11}, M_{12}, J_3, A_5, A_6$ being 7′-group in the lists of alternating and sporadic simple groups. In the lists of Lie groups $G(q)$, if $(q^6 - 1) \mid |G(q)|$, then $G(q)$ can not be 7′-group since $m^6 \equiv 1$ (mod7) for any positive integer number m. Therefore, we need only to consider the Lie groups: $A_n(q)$, $n = 1, 2, 3, 4$, $B_2(q)$, ${}^2A_n(q)$, $n = 2, 3, 4$, ${}^2F_4(2^{2m+1})$, ${}^2B_2(2^{2m+1})$, and ${}^2G_2(3^{2m+1})$, $m \geq 1$.

If $A_1(q)$ is a 7′-group, i.e., $(7, |A_1(q)|) = (7, q^2(q^2 - 1)d^{-1}) = 1$, here $d = (2, q - 1)$, which is equivalent to $q \not\equiv 0, 1$ and -1(mod 7). If $B_2(q)$ is a 7′-group, then $(7, |B_2(q)|) = (7, q^4(q^4 - 1)(q^2 - 1)) = 1$, this is equivalent to $q \not\equiv 0, 1$ and -1(mod 7) for $q^4 \equiv 1$ mod7) if and only if $q^2 \equiv 1$ (mod7). These are the list of (2). By the same argument, we get the lists of (3), (4) and (5). But for ${}^2G_2(3^{2m+1})$ since $|{}^2G_2(q)| = q^3(q^3 - 1)(q - 1)$, $q = 3^{2m+1}$ and $q^3 + 1 = 3^{3(2m+1)} + 1 \equiv 0$(mod7) for all nonnegative integer number m, ${}^2G_2(q)$ can not be a 7′-group. The proof is completed. □

Lemma 2 *Let* $F(q)$ *be a field of* q *elements, then* $(-7)^{1/2}$ *is an element of* $F(q)$ *if and only if* $q^3 \equiv 1$(mod7).

Proof See the proof of lemma 6.5 of [5, p. 173].

The following lemma is clear.

Lemma 3 If H is a proper non-abelian simple section of an inner 7-closed simple group G, then H must be 7′-group.

Proof of the Theorem Firstly, one can easily check that the listed groups in the theorem are really inner 7-closed groups by Diekson theorem [6, p. 213]. Conversely, let G be an inner 7-closed simple group, we shall show that G must be isomorphic to one of the listed groups of the theorem with the help of the classification theorem of finite simple groups.

(2.1) G is not isomorphic to any one of the list of alternating groups and sporadic groups. Looking up the ATLS table [7] (especially for the items which the maximal subgroups were listed in), it can be seen that such a simple group is either 7′-group or having a proper section $A_1(7)$ or A_7, and A_7 has also a proper section $A_1(7)$. Hence all of the alternating groups $A_n (n \geq 5)$ and spradic groups are not inner 7-closed group by lemma 3.

(2.2) G is not of type: $E_6(q)$, $E_7(q)$, $E_8(q)$, ${}^3D_4(q)$, ${}^2E_6(q)$.

Since the listed groups of (2.2) have Levi subgroup (see [8]) $A_5(q), A_6(q), A_7(q), B_3(q), A_1(q^3)$ and ${}^2D_4(q)$, respectively. These subgroups are not 7′-group by lemma 1 and are simple groups. It follows that the listed groups of (2.2) can not be inner 7-closed group by lemma 3, and (2.2) holds.

(2.3) If G is of type $A_n(q)$, $q = p^f$, then G must be one of the listed groups of the theorem.

Suppose $G \cong A_n(q)$ for some q and n. It is known that $A_n(q)$ has a proper subgroup $A_{n-1}(q)$, and $A_{n-1}(q)$ is simple when $n \geq 3$ or $q \geq 4$. By lemma 1 and lemma 3, it follows that $n - 1 \leq 4$. Thus we need to consider the family $A_n(q)$, $n = 1, 2, 3, 4, 5$.

(2.3.1) Suppose that $G \cong A_1(q)$, Since $7 \mid |A_1(q)| = q^2(q^2-1)d^{-1}$, $d=(2, q-1)$, $q \equiv 0, 1$ or $-1 \pmod 7$. If $q \equiv 0 \pmod 7$, $q=7^f$, then $f=1$ for $A_1(7)$ is a subgroup of $A_1(7^f)$. While $A_1(7)$ belongs to the listed groups of the theorem. Since the Broel subgroup B of $A_1(q)$ is a Frobenius group with kernel K of order q and complement H of order $(q-1)d^{-1}$. If B is 7-closed, then $q \not\equiv 1 \pmod 7$. Therefore, if $A_1(q)$ is inner 7-closed group, then $q \not\equiv 1 \pmod 7$. Thus one can assume that $q=p^f \equiv -1 \pmod 7$. If $p \equiv -1 \pmod 7$, then $p \geq 13$, $7 \mid |A_1(p)| = p(p^2-1)2^{-1}$, $A_1(p)$ is simple and is a subgroup of $A_1(q)$, this forces that $f=1$. $A_1(p)$, $p \equiv -1 \pmod 7$, is an inner 7-closed group and is contained in the listed groups of the theorem. If $p+1 \not\equiv 0 \pmod 7$ and $q=p^f \equiv -1 \pmod 7$, this case occurs if and only if $p \equiv 3$ or $5 \pmod 7$ and $3 \mid f$. It follows that $A_1(p^3)$ is a subgroup of $A_1(q)$ by Dickson theorem [6, p. 213], and $7 \mid |A_1(p^3)| = p^3(p^6-1)2^{-1}$, $A_1(p^3)$ is a simple group. This forces that $f=3$ by lemma 3. $A_1(p^3)$, $p \equiv 3$ or $5 \pmod 7$, is an inner 7-closed group and is also contained in the listed groups of the theorem.

(2.3.2) G is not of the type $A_n(q)$, $n=2, 3, 4, 5$, except $A_2(2)$. $|A_2(q)| = q^3(q^3-1)(q^2-1)d^{-1}$, $d=(3, q-1)$. If $q^3 \equiv 1 \pmod 7$ and $q>2$, then $A_2(q)$, q odd, has a proper subgroup $A_1(7)$ from [5] for odd q and from $A_2(2) \cong A_1(7)$ for $q=2^f$, hence $A_2(q)$ can not be an inner 7-closed group except $A_2(2)$ in case $q^3 \equiv 1 \pmod 7$. Since $A_1(q)$ is a subgroup of $A_2(q)$ and is a simple group if $q>3$, but is not $7'$-group if $q \equiv 0$ or $\pm 1 \pmod 7$ by lemma 1. It follows that $A_2(q)$ can not be inner 7-closed group if $q \equiv 0$ or $\pm 1 \pmod 7$ and $q>3$. While $A_2(3)$ is $7'$-group. Hence (2.3.2) holds for $n=2$.

It is known that $A_2(q)$ is a proper subgroup of $A_3(q)$ and $A_4(q)$, $A_3(q)$ and $A_4(q)$ are $7'$-group if $A_2(q)$ is $7'$-group by lemma 1 and, therefore, are not inner 7-closed group by lemma 3. Since SL $(3, q^2)$ is a subgroup of SL $(6, q)$, $A_2(q^2)$ is a proper section of $A_5(q)$ and not $7'$-group by lemma 1, hence $A_5(q)$ is not inner 7-closed group. Therefore, (2.3.2) holds, and (2.3) holds.

(2.4) G is not of the type $B_n(q)$, $n \geq 2$.

Assume that $G \cong B_n(q)$ for some n and $q=p^f$. By lemma 1 and 3, $n=2$ or 3 for $B_{n-1}(q)$ is a subgroup of $B_n(q)$. $B_2(q)$ has a Levi subgroup $L=A_1(q)B_1(q)$, $A_1(q)$ $(q>3)$ is simple group. $B_2(q)$ $(q>3)$ is $7'$-group if $A_1(q)$ is $7'$-group by lemma 1, hence $B_2(q)$ can not be inner 7-closed group as it has the proper section $A_1(q)$ $(q>3)$. $B_2(2)$ and $B_2(3)$ are $7'$-groups. Now, one has $G \cong B_3(q)$ for some q. Since $G_2(q)$ is a subgroup of $B_3(q)$ (see [9, §1.4], where $B_3(q) \cong P\Omega_7(q)$) and is not $7'$-group by lemma 1, hence $B_3(q)$ can not be inner 7-closed group. A contradiction, and (2.4) holds.

(2.5) G is not of type $C_n(q)$, $n \geq 3$.

Since the subgroup $C_{n-1}(q)$ $(n>3)$ of $C_n(q)$ is not $7'$-group, $C_n(q)$ $(n>3)$ is not inner 7-closed group by lemma 3. $Sp(2, q^3)$ is subgroup of $Sp(6, q)$ by [6, p. 228], it follows that $C_1(q^3)$ is a subgroup of $C_3(q)$, $C_1(q^3) = A_1(q^3)$ is not $7'$-group by lemma 1, hence $C_3(q)$ is not inner 7-closed group, and (2.5) holds.

(2.6) G is not of type $D_n(q)$, $n>3$, $G_2(q)$, $q=p^f$.

Similarly, we need only to consider the case $n=4$. It is known that $D_4(q)$ has a graph automorphism u of period, the fixed subgroup of u is the Lie group $G_2(q)$. It follows that $D_4(q)$ can not be inner 7-closed group for its subgroup $G_2(q)$ is not $7'$-group by lemma 1. Since $A_1(q)$ is a proper section of $G_2(p)$ (see[9, Theorem A] for odd prime p), hence $G_2(p)$ is not inner 7-closed group. Obviously, $G_2(p)$ is a subgroup of $G_2(q)$, $q = p^f$, and $G_2(q)$ can not be inner 7-closed group for its subgroup $G_2(p)$ is not $7'$-group, and (2.6) holds.

(2.7) G is not of type ${}^2A_n(q)$, $n > 1$.

Similary, by Lemma 1 and 3, we need to consider the case $n < 6$. Since ${}^2A_2(q)$ is a subgroup of ${}^2A_3(q)$ and ${}^2A_4(q)$, if ${}^2A_2(q)$ is $7'$-group, then ${}^2A_3(q)$ and ${}^2A_4(q)$ is also $7'$-group, hence ${}^2A_n(q)$ $(n = 3, 4)$ can not be inner 7-closed group. Since the Levi subgroup $A_2(q^2)$ of ${}^2A_5(q)$ is not $7'$-group, ${}^2A_5(q)$ can not be inner 7-closed group. If $n = 2$, one can assume that $7 \mid |{}^2A_2(q)| = q^3(q^3+1)(q^2-1)d^{-1}$, $d = (3, q+1)$, i.e., $q \equiv 0$ or $q^2 \equiv 1$ or $q^3 \equiv -1 \pmod 7$. If $q \equiv 0 \pmod 7$, i.e., $q = 7^f$. Since ${}^2A_2(7) = U_3(7)$ has a maximal subgroup $A_1(7).2$ (see [7, p. 66]), ${}^2A_2(7)$ is a subgroup of ${}^2A_2(q)$, thus ${}^2A_2(q)$ can not be inner 7-closed group. If $q^2 \equiv 1 \pmod 7$, the Borel subgroup B of ${}^2A_2(q)$ is the semidirect product of P of order q^3 and H of order $(q^2-1)/d$, $C_P(H) = 1$ implies that B can not be 7-closed group, hence ${}^2A_2(q)$ can not be inner 7-closed group. Now, assume that $q^3 \equiv -1 \pmod 7$, then q is odd. We show that $A_1(7)$ is a section of ${}^2A_2(q)$. To do this, let us recall the definition of ${}^2A_2(q)$. Let $SL(3, q^2)$ be the three dementional special linear group in matrix form with entries in a finite field $F(q^2)$. By classical notations, the special unitary group

$$SU(3, q) = \{x \in SL(3, q^2) \mid X^{-1} = (X^u)^t\}$$

(see [10, p. 466]), where "t" denotes transpose and X^u is the matrix abtained from X by applying the field automorphism $a^u = a^q$ of $F(q^2)$ to each entries. Thus ${}^2A_2(q) = PSU(3, q)$ is the factor group of $SU(3, q)$ modulo its center and $|SU(3, q) : {}^2A_2(q)| < 4$. This implies that $A_1(7)$ is a section of ${}^2A_2(q)$ in case $A_1(7)$ is a subgroup of $SU(3, q)$, so it suffices to show that $A_1(7)$ is a subgroup of $SU(3, q)$. Let

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad Y = \begin{bmatrix} r & 2^{-1} & 2^{-1} \\ r & -2^{-1} & 2^{-1} \\ 0 & r+2^{-1} & r+2^{-1} \end{bmatrix},$$

where $r \in F(q^2)$ and $2r^2 + r + 1 = 0$. Note that such an element r do exist in $F(q^2)$ with odd q, i.e., the equation has at least one solution in the field. That occurs if and only if the element $(-7)^{1/2} \in F(q^2)$, and if and only if $q^6 \equiv 1 \pmod 7$ by Lemma 2. Clearly, this is always holds. After calculation, it is seen that $X^2 = Y^7 = (XY)^3 = (XY^3)^4 = 1$ and these relations generate $A_1(7)$ by [11, p. 303]. Clearly, $X \in SU(3, q)$, $Y \in SL(3, q^2)$. Now we need to show that $Y \in SU(3, q)$, i.e, $Y(Y^u)^t = 1$. This equation holds if and only if $r^{q+1} = 2^{-1}$. Let $x = (-7)^{1/2}$ then $x^2 = -7 \in F(q^2)$, $(x^{q-1})^2 = (-7)^{q-1} = 1$, hence $x^{q-1} = 1$ or -1. If $x^{q-1} = 1$ then $x \in F(q)$ and $q^3 \equiv 1 \pmod 7$ by Lemma 2, this is contray to that $q^3 \equiv -1 \pmod 7$. Hence $x^{q-1} = -1$ and $x^q = -x$, let $r = (x-1)/4$, then $2r^2 + r + 1 = 0$ and $r^{q+1} = 2^{-1}$. Therefore, $Y \in SU(3, q)$ and $A_1(7)$ is a subgroup of ${}^2A_2(q)$. Summaring the above argument, (2.7) holds.

(2.8) G is not of type ${}^2B_2(q)$, ${}^2F_4(q)$, $q = 2^{2m+1}$.

$|{}^2B_2(q)| = q^2(q^2+1)(q-1)$, $q = 2^{2m+1}$, $(q^2(q^2+1), 7) = 1$, it follows that only the case "$q \equiv 1 \pmod 7$" need to be considered. In this case, the Borel subgroup of ${}^2B_2(q)$ is a Frobennius group with kernel K of order q^2 and complement H of order $q-1$. Thus B is not 7-closed, and so ${}^2B_2(q)$ is not inner 7-closed group. ${}^2B_2(q)$ is a Levi subgroup of ${}^2F_4(q)$, if ${}^2B_2(q)$ is $7'$-group, then