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Note on Magic Squares and Magic Cubes on Abelian Groups*

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Abstract An attempt is made to construct magic squares and magic cubes on a given abelian group.

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0. Introduction

The original magic squares, e.g.,

$$\begin{matrix} 6 & 1 & 8 \\ 7 & 5 & 3 \\ 2 & 9 & 4 \end{matrix} \quad \text{and} \quad \begin{matrix} 1 & 12 & 8 & 13 \\ 15 & 6 & 10 & 3 \\ 14 & 7 & 11 & 2 \\ 4 & 9 & 5 & 16 \end{matrix}$$

are defined as an arrangement of the first n^2 positive integers into a square matrix so that the sum of each column, row, and diagonal numbers add up to the magic number $n(n^2+1)/2$.

It does not take much to realize that if each entry of the magic square is decreased by 1 the resulting square matrix is again a magic square with magic number $n(n^2-1)/2$. Thus one can think of the old-fashioned magic squares as magic squares on the cyclic groups of order n^2.

We deffine a magic square of order n on a group G (which is necessarily of order n^2) as an arrangement of the n^2 elements of G into a square matrix so that each row, column, or diagonal has its product (or sum depending on the operation of G) equal to the same group element. For abelian groups, there is no ambiguity in multiplying diagonal elements. In the case of nonabelian groups, some convension is needed.

1. Magic squares on cyclic groups

It is well known that magic squares of all orders exist. This fact may be restated as

Proposition 1. *Magic squares on cyclic groups of order n^2 exist for all positive integer $n \geq 3$.*

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Notice that we leave the old convention that an $n \times n$ square is of order n. The old magic square of order 3 is now the magic square on the cyclic group of order 9, or $\mathbf{Z}_9$.

2 Magic squares on groups of order p^2

Our purpose is to go after the following

Theorem. *Magic squares on all abelian groups of order n^2 exist, for all positive integer $n \geq 3$.*

Several special cases are obvious:

Proposition 2 *Magic squares on abelian groups of order p^2 exist for all odd prime p.*

Groups of order p^2 are either cyclic or elementary abelian. We have already seen that magic squares on cyclic groups exist. To prove this proposition for elementary abelian groups, we list the group elements like the "time table" in the natural order, then it is easy to see that the rows, columns and the diagonals have the same sum.

3 Magic squares on elementary abelian groups

From the consideration of groups of order p^2, we see that this method easily extends to all elementary abelian groups of order p^{2n}. For example, a magic square on $\mathbf{Z}_2^4$, the elementary abelian 2-group of order 16:

(0, 0, 0, 0)	(0, 1, 0, 0)	(1, 0, 0, 0)	(1, 1, 0, 0)
(0, 0, 1, 0)	(0, 1, 1, 0)	(1, 0, 1, 0)	(1, 1, 1, 0)
(0, 0, 0, 1)	(0, 1, 0, 1)	(1, 0, 0, 1)	(1, 1, 0, 1)
(0, 0, 1, 1)	(0, 1, 1, 1)	(1, 0, 1, 1)	(1, 1, 1, 1)

It is easy to see that this is a magic square since there is an even number of 1's occuring on each coordinate of each row, column, and diagonal. Therefore, each row, column, and diagonal adds up to (0, 0, 0, 0). We have the following

Proposition 3 *For any prime p, and any positive integer n, magicc squares on elementary abelian p-groups of order p^{2n} exist.*

4 Automorphisms of magic squares

By an automorphism of a magic square, we mean a bijection of the magic square onto itself that preserves its magic character, i.e., the result is again a magic square; e.g., in the case of a 4×4 magic square on $\mathbf{Z}_{16}$,

0	11	7	12
14	5	9	2
13	6	10	1
3	8	4	15

It is easily seen that the bijection of exchanging both the first two rows and columns is an automorphism. Now for magic squares on elementary abelian p-groups, exchanging rows and columns preserves its magic character. We also verify easily that exchanging any two rows or columns preserves the magic character. Following [2], we call the automorphism group of a magic square its magic group. We have the following

Theorem. *The magic group of a magic square on* $\mathbf{Z}_p^{2n}$ *contains the group* $S_{p^n} \times S_{p^n}$.

5. Magic cubes on abelian groups

In [1], a magic cube of order 3 is given: technically, that is not really a magic cube because the diagonals on each face or layer do not add up. According to the previous discussion the old-fashioned "magic cube of order 3" should be called a magic cube on $\mathbf{Z}_{27}$, the cyclic group of order 27, which, we shall show, does not exist. Then magic cubes on other abelian groups, especially elementary abelian p-groups, are plentiful.

Example (Nonexxistence of magic cubes on $\mathbf{Z}_{27}$) Assuming there were a magic cube on $\mathbf{Z}_{27}$, and let the magic sum be M. Let one of the faces be

$$\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}$$

Since sum of each column, row, and diagonal must be M, we have $a+e+i=c+e+g=a+d+g=c+f+i$, so the four corners $a+c+g+i=M-2e=M-(d+f)$, which implies that $2e=d+f$, adding e to both sides gives $3e=M$, so e must be $M/3$. The same argument goes toward the opposite face, so the middle element must also be $M/3$, which contradicts the fact that the 27 elements are distinct.

Automorphisms By an automorphism (see [3]) of a magic cube, we mean a bijection of the magic cube which preserves the magic character of the cube. We observe that exchanging the layers of the cube in any manner preverves the magic character of the cube. Also, any automorphism of the group induces an automorphism of the cube, and we have the following

Theorem. *The automorphism group of a magic cube on an elementary* p*-group of order* p^{3n} *contains as subgroups the automorphism group and* $S_{p^n} \times S_{p^n} \times S_{p^n}$, *where* S_{p^n} *is the symmetric group on* p^n *symbols.*

References

[1] W. S. Andrews, Magic Squares & Cubes, Dover, 1960.

[2] H. S. H. Sun and I. Wen, On magic groups and their applications to 4×4 magic squares, Sichuan J. of Math., 1990.