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紧对称空间上 Riesz 平均的强逼近

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摘 要

本文在紧对称空间中给出了全测度集上 Riesz 平均强逼近的阶.

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Strong Approximation by Riesz Means on Compact Symmetric Spaces*

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Abstract The rates of strong approximation by Riesz means on compact symmetric spaces are established.

Keywords strong approximation, Riesz means, symmetric space.

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Let M be a d-dimensional $(d > 1)$ compact Riemannian symmetric space of rank one. By $L^p(M)$, $1 \le p < \infty$, we denote the space of (the equivalence classes of) p-integrable functions on M with the norm

$$\| f \|_p := \left\{ \int |f(x)|^p dm(x) \right\}^{1/p},$$

dm being the invariant normalized Riemannian measure on M.

We know that with the exception of the circle the compact symmetric spaces of rank one coincide with the compact two-point homogeneous spaces (see [2] p. 355). H. C. Wang[6] has shown that M is one of the following five types, the sphere Σ_d, the real projective space $P^d(\mathbb{R})$, the complex projective space $P^d(C)$, the quaternionic projective space $P^d(H)$, and the Cayley elliptic plane $P^{16}(Cay)$, where d denote the dimension of M as a manifold over the reals.

Let Δ be the Laplace-Beltrami operator on M and let

$$L^2 = \bigoplus_{n=0}^{\infty} \mathbf{H}_n$$

be the decomposition of the space L^2 in a direct orthogonal sum of finite dimensional subspaces $\mathbf{H}_n$. The subspaces $\mathbf{H}_n$ are the eigensubspaces of Δ corresponding to the eigenvalues $-\lambda_n^2 \equiv -\lambda_n^2(\alpha, \beta) = -n(n+\alpha+\beta+1)$:

$$\mathbf{H}_n = \{ f \in C^\infty(M) : \Delta f = -\lambda_n^2(\alpha, \beta) f \},$$

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where $\alpha=(d-2)/2$, $\beta=\alpha$ for Σ_d and $\beta=-\frac{1}{2}$, 0, 1 and 3 for $P^d(\mathbb{R})$, $P^d(C)$, $P^d(\mathbf{H})$ and $P^{16}(Cay)$, respectively.

Each $\mathbf{H}_n$ contains only one zonal spherical function $\varphi_n(x)=\varphi_n^*(r)$, which depends only on $r=\rho(x_0,x)$ (ρ is the geodesic distance on M), and is normalized by the condition $\varphi_n(x_0)=\varphi_n^*(0)=1$. Let $L=\sup\{\rho(x,y): x,y\in M\}$ be the diameter of M. It is known that the functions $\varphi_n^*(r)$ are normalized Jacobi polynomials

$$\varphi_n^*(r)=P_n^{(\alpha,\beta)}\left(\cos\frac{\pi y}{L}\right)/P_n^{(\alpha,\beta)}(1).$$

The orthogonal projection $Y_n: L^2\to\mathbf{H}_n$ is given by the formula

$$Y_n(f;x)=\|\varphi\|_2^2\int_M f(y)\varphi_n\left(\cos\left(\frac{\pi}{L}\rho(x,y)\right)\right)dm(y).$$

Consequently, for a given function f on M one has the eigenfunction expansion

$$f\sim\sum_{n=0}^{\infty}Y_n(f). \tag{1}$$

The Riesz means of the series of f in (1) are defined by

$$S_R^{\kappa,\delta}(f;x)=\sum_{\lambda_k<R}\left(1-\left(\frac{\lambda_k}{R}\right)^{\kappa}\right)^{\delta}Y_k(f;x),\ \kappa, R>0. \tag{2}$$

The index δ is restricted to be positive in usual arguments. As δ increases, the convergence properties of these operators improve. We refer to [1]. Considering strong convergence, the index can be extend to be negative. Indeed, by following straight forward modifications of the proof of the corresponding result in [5], we have

Theorem 1 *Write*

$$S^{\kappa,\delta}(f;x):=\sup_{R>1}\left\{\frac{1}{R}\int_1^R\left|S_r^{\kappa,\delta}(f;x)\right|^2dr\right\}^{1/2}.$$

Let $\delta>d(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}$. For $f\in L^p(M)$, $1<p\leq 2$

$$\|S^{\kappa,\delta}(f)\|_p\leq \mathrm{const}_{\kappa,\delta,p}\|f\|_p, \tag{3}$$

where the "const" is a constant depending only on the indicated subindices.

By standard density arguments, we have the strong summability for these operators from (3), i.e.,

$$\lim_{R\to\infty}\frac{1}{R}\int_1^R\left|S_r^{\kappa,\delta}(f;x)-f(x)\right|^2dr=0 \tag{4}$$

almost everywhere on M for $\delta>d(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}$ and for $f\in L^p(M)$, $1<p\leq 2$.

We are interesting in establishing the quantitative form of the convergence in (4). That is to say, we would like to study the strong approximation by Riesz means on sets of full measure.

Let $\sigma>0$, and let $f\in L^2(M)$. If there exists a function $g\in L^2(M)$ such that $Y_k(g;\cdot)=\lambda_k^{\sigma}Y_k(f;\cdot)$, $\forall k\in N_0$, we call g the Riesz derivative of f of order σ and write $f^{\{\sigma\}}=g$, we see that $f^{\{\sigma\}}$

is uniquely determined by f. Let D^{σ}, $\sigma > 0$, denote the Riesz space of order σ, consisting of those elements $f \in L^2(M)$ for which

$$\| f \|_{2,\sigma} \equiv \| f \|_{D^{\sigma}} = \left\{ \| Y_0 f \|_2^2 + \sum_1^{\infty} \lambda_k^{2\sigma} \| Y_k f \|_2^2 \right\}^{1/2} < \infty.$$

With these notations we state the main result of this note as follows

Theorem 2 *Let $f \in D^{\sigma}$, $0 < \sigma \le \kappa$, and let $\delta > -\frac{1}{2}$. For almost all x in M*

$$\frac{1}{R}\int_1^R \left| S_r^{\kappa,\delta}(f;x) - f(x) \right|^2 dr = \begin{cases} O_x\left(\frac{1}{R^{2\sigma}}\right), & \text{if } 0 < \sigma < \frac{1}{2}, \\ O_x\left(\frac{\log R}{R}\right), & \text{if } \sigma = \frac{1}{2}, \\ O_x\left(\frac{1}{R}\right), & \text{if } \sigma > \frac{1}{2}. \end{cases} \tag{5}$$

Remark If $\delta > 0$, we have already proved for $f \in D^{\sigma}$, $0 < \sigma \le \kappa$ that (see [3, 4])

$$\left| S_R^{\kappa,\delta}(f;x) - f(x) \right| = O_x\left(\frac{1}{R^{\sigma}}\right) \tag{6}$$

holds almost everywhere on M. From (6) we can directly deduce the desired results. The key step is to handle the case $-\frac{1}{2} < \delta \le 0$.

Proof of Theorem 2 We focus on the case $-\frac{1}{2} < \delta \le 0$. We introduce the Littlewood-Paley g function

$$g_{\sigma}^{\delta}(f;x) := \left\{ \int_0^{\infty} r^{2\sigma-1} \left| S_r^{\kappa,\delta}(f;x) - S_r^{\kappa,\delta+1}(f;x) \right|^2 dr \right\}^{1/2}.$$

The function g_{σ}^{δ} is bounded on D^{σ}, i.e., for $\delta > -\frac{1}{2}$, $f \in D^{\sigma}$, $0 < \sigma < \kappa$,

$$\| g_{\sigma}^{\delta}(f) \|_2 \le \mathrm{const}_{\delta,\sigma,\kappa} \| f \|_{2,\sigma} \tag{7}$$

In fact, by making use of the Parseval formula and Fubini's theorem, it follows that

$$\begin{aligned} \| g_{\sigma}^{\delta}(f) \|_2^2 &= \int_M \int_0^{\infty} r^{2\sigma-1} \left| S_r^{\kappa,\delta}(f;x) - S_r^{\kappa,\delta+1}(f;x) \right|^2 dr\, dm(x) \\ &= \int_0^{\infty} \sum_{\lambda_k < r} r^{2\sigma-1} \left| \left(1 - \left(\frac{\lambda_k}{r}\right)^{\kappa}\right)^{\delta+1} - \left(1 - \left(\frac{\lambda_k}{r}\right)^{\kappa}\right)^{\delta} \right|^2 \| Y_k(f) \|_2^2 dr \\ &= \sum_{k=0}^{\infty} \int_{\lambda_k}^{\infty} \left(1 - \left(\frac{\lambda_k}{r}\right)^{\kappa}\right)^{2\delta} \frac{\lambda_k^{2\kappa}}{r^{2\kappa-2\sigma+1}} \| Y_k f \|_2^2 dr \\ &= \sum_{k=0}^{\infty} \lambda_k^{2\sigma} \| Y_k f \|_2^2 \int_0^1 \kappa^{-1} (1-s)^{2\delta} s^{1-2\sigma/\kappa} ds \\ &\le \mathrm{const}_{\delta,\sigma,\kappa} \| f \|_{2,\sigma}^2 \end{aligned}$$

obtaining the required estimate (7).

Passing to prove (5). For $0 < \sigma \le \frac{1}{2}$,

$$
\begin{aligned}
&\left\{ \frac{R^{2\sigma}}{R} \int_1^R \left| S_r^{\kappa, \delta}(f; x) - S_r^{\kappa, \delta+1}(f; x) \right|^2 dr \right\}^{1/2} \\
&\le \left\{ \int_1^R r^{2\sigma-1} \left| S_r^{\kappa, \delta}(f; x) - S_r^{\kappa, \delta+1}(f; x) \right|^2 dr \right\}^{1/2} \\
&\le g_\sigma^\delta(f; x).
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
&\left\| \sup_{R>1} \left\{ \frac{R^{2\sigma}}{R} \int_1^R \left| S_r^{\kappa, \delta}(f; \bullet) - S_r^{\kappa, \delta+1}(f; \bullet) \right|^2 dr \right\}^{1/2} \right\|_2 \\
&\le \| g_\sigma^\delta(f) \|_2 \le const_{\delta, \sigma, \kappa} \| f \|_{2, \sigma}
\end{aligned}
$$

which gives

$$
\frac{1}{R} \int_1^R \left| S_r^{\kappa, \delta}(f; x) - S_r^{\kappa, \delta+1}(f; x) \right|^2 dr = O_x\left(\frac{1}{R^{2\sigma}} \right)
$$

almost everywhere on M for $f \in D^\sigma$ and $0 < \sigma \le \frac{1}{2}$. If $f \in D^\sigma$, $\sigma > \frac{1}{2}$, then $f \in D^{\frac{1}{2}}$. We also have

$$
\frac{1}{W\ TFZ〗\ R} \int_1^R \left| S_r^{\kappa, \delta}(f; x) - S_r^{\kappa, \delta+1}(f; x) \right|^2 dr = O_x\left(\frac{1}{R} \right), \text{ a. e.} \tag{8}
$$

On the other hand, it is easy to check that

$$
\begin{aligned}
&\frac{1}{R} \int_1^R \left| S_r^{\kappa, \delta}(f; x) - f(x) \right|^2 dr \\
&\le \frac{2}{R} \int_1^R \left| S_r^{\kappa, \delta}(f; x) - S_r^{\kappa, \delta+1}(f; x) \right|^2 dr \\
&\quad + \frac{2}{R} \int_1^R \left| S_r^{\kappa, \delta+1}(f; x) - f(x) \right|^2 dr.
\end{aligned}
$$

By this inequality, (6) and (8), we finally get

$$
\frac{1}{R} \int_1^R \left| S_r^{\kappa, \delta}(f; x) - f(x) \right|^2 dr =
\begin{cases}
O_x\left(\frac{1}{R^{2\sigma}}\right), & \text{if } 0 < \sigma < \frac{1}{2}, \\
O_x\left(\frac{\log R}{R}\right), & \text{if } \sigma = \frac{1}{2}, \\
O_x\left(\frac{1}{R}\right), & \text{if } \sigma > \frac{1}{2}
\end{cases}
$$

hold almost everywhere on M.