$-\frac{1}{2}\langle Lu, u\rangle + F(u), u \in X.$

Proof Let $X_0 = N(L)$, $a(u) = -\frac{1}{2}\langle Lu, u\rangle$, $g(u) = F(u)$ for $u \in X$. By the definition of a and (L_3), we have

$$ta(u) + (1-t)a(w) - a(tu + (1-t)w) = t(1-t)a(u-w) \geq 0,$$

for all $u, w \in X$ and $t \in [0, 1]$. Hence a is convex. From the continuity of L, we obtain the continuity of a. By Theorem 1, φ has a minimum u in X. Hence $\varphi'(u) = 0$. But $\varphi'(u) = -Lu + F'(u)$. Therefore $Lu = F'(u)$. Hence the equation (4) has at least one solution $u \in X$ which minimizes the action φ

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自反 Banach 空间上的半强制单调变分问题

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摘 要

仅用变分法的经典定理, 研究了自反 Banach 空间上的半强制单调变分问题. 获得了自反 Banach 空间上半线性方程解的存在性.

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Semi-Coercive Monotone Variational Problems on Reflexive Banach Spaces*

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Abstract Using only chassical theorems of calculus of variations, we study semi-coercive monotone variational problems on reflexive Banach spaces. We obtain the existence of solutions for semilinear equations on reflexive Banach spaces.

Keywords minimizing sequence, minimum, coercive, lower semicontinuous, convex function, semilinear equation.

Classification AMS(1991) 49J27, 47H15/CCL O176.3

J. Mawhin[3] studied semi-coercive monotone variational problems on Hilbert spaces with the direct method of the calculus of variations and the basic principle of convex analysis. Motivated by Mawhin[3], we study the corresponding problems on reflexive Banach spaces using only classical theorems of the variational calculus. Then we apply the abstract results to semilinear equations on reflexive Banach spaces and obtain an existence theorem of solutions.

Let X be a real reflexive Banach space with norm $\|\cdot\|$, $a: X\to R\cup\{+\infty\}$ a proper convex and lower semicontinuous function, $g: X\to R$ a convex lower semicontinuous function, $f=a+g$. Let X^* be the dual space of X, $\langle\cdot,\cdot\rangle: X^*\times X\to R$ the canonical bilinear mapping. Assume that X_0 and X_1 are closed subspaces of X such that $X=X_0\oplus X_1$, for $u\in X$, write $u=\bar{u}+\tilde{u}$ with $\bar{u}\in X_0$, $\tilde{u}\in X_1$. In a way similar to [3], we shall say that the function a is X_1-coercive if there exists a function $\alpha: [0,+\infty]\to R$ such that $\alpha(t)\to+\infty$ as $t\to+\infty$ and

$$a(u)\geq\alpha(\|\tilde{u}\|)\,\|\tilde{u}\| \tag{1}$$

for all $u\in X$.

Theorem 1 *Assume that* a *is* X_1*-coercive and*

$$g(u)\to+\infty \text{ as } \|u\|\to\infty \text{ in } X_0. \tag{2}$$

Then f *has a minimum in* X.

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Proof From (2) and Theorem 1.1 in [4], g has a minimum b in X_0, that is

$$g(b+h) \geq g(b) + \langle 0, h \rangle$$

for all $h \in X_0$. By the Hahn-Banach theorem (see e.g. [2]), there exists $a_0 \in X^*$ such that $a_0 |X_0 = 0$ and

$$g(b+h) \geq g(b) + \langle a_0, h \rangle \tag{3}$$

for all $h \in X$.

Suppose that (u_k) is a minimizing sequence of f. Without loss of generality we may assume that $(f(u_k))$ is decreasing. By (1) and (3), we have

$$\begin{aligned} f(u_k) &\geq \alpha(\| \tilde{u}_k \|) \| \tilde{u}_k \| + g(b) + \langle a_0, u_k - b \rangle \\ &= \alpha(\| \tilde{u}_k \|) \| \tilde{u}_k \| + g(b) + \langle a_0, \tilde{u}_k \rangle \\ &\geq (\alpha(\| \tilde{u}_k \|) - a_0 \|) \| \tilde{u}_k - + g(b)) \end{aligned}$$

for all $k \in N$. From the boundedness of $(f(\tilde{u}_k))$, one obtains that $(\tilde{u}_k)$ is bounded, Without loss of generality, we may assume that $(\tilde{u}_k)$ weakly converges to some $\tilde{u} \in X_1$. By Mazur's lemma (see e.g. [7]), there exist positive integer $N_k \geq k$ and nonnegative constant λ_n^k for every $k \in N$ and $k \leq n \leq N_k$, such that $\sum_{n=k}^{N_k} \lambda_n^k = 1$ for $k \in N$ and $\sum_{n=k}^{N_k} \lambda_n^k \tilde{u}_n \to \tilde{u}$ as $k \to \infty$. Let $v_k = \sum_{n=k}^{N_k} \lambda_n^k u_n$ for $k \in N$. By the convexity of f and the decreasing property of $(f(u_k))$, one has $f(v_k) \leq \sum_{n=k}^{N_k} \lambda_n^k f(u_n) \leq f(u_k)$ for all $k \in N$. Hence (v_k) is a minimizing sequence of f. From Corollary 4.3.15 in [1], g is continuous. Hence $g(-\tilde{v}_k) \to g(-\tilde{u})$ as $k \to \infty$, thus $(g(-\tilde{v}_k))$ is bounded. By the convexity of g, one has $f(v_k) \geq g(v_k) \geq 2g(\frac{\bar{v}_k}{2}) - g(-\bar{v}_k)$ for all $k \in N$. From (2), we obtain that $(\bar{v}_k)$ is bounded. Hence (v_k) is bounded. By Theorem 1.1 in [4], f has a minimum in X.

Remark 1 Theorem 1 is an extension on reflexive Banach spaces of Theorem 1 in [3].

In [5] and [6], we study respectively the existence of solutions and nontrival solutions for semilinear equations on reflexive Banach spaces with the duality technique and convex and nonsmooth analysis.

Now we shall study the existence of solutions for semilinear equations on reflexive Banach spaces with Theorem 1. Consider the semilinear equation

$$Lu = F'(u). \tag{4}$$

Theorem 2 Suppose that $L: X \to X^*$ is a continuous linear operator such that

(L_1) $\langle Lu, w \rangle = \langle Lw, u \rangle$ for all $u, w \in X$.

(L_2) the nullspace $N(L)$ of L has a complementary subspace X_1 in X, that is, $X_1 \oplus N(L) = X$.

(L_3) There is a positive constant α such that $\langle Lu, u \rangle \leq -\alpha \| u \|^2$ for all $u \in X_1$.

Assume that $F: X \to R$ is a Gateaux differentiable, lower semicontinuous and convex function satisfying the condition

$$F(w) \to +\infty \text{ if } \| w \| \to \infty \text{ in } N(L). \tag{5}$$

Then equation (4) has at least one solution $u \in X$ which minimizes the action φ defined by $\varphi(\mu) =$