By Radon-Nikodym theorem, there exists a non-negative measurable function g such that $\upsilon(A) = \int_A g d\mu$ whenever A belongs to S. Then for each $n \in N$, It is satisfied that $\upsilon_n(A) = \upsilon(A \cap E_n) = \int_{A \cap E_n} g d\mu$. We first show that $\int_{A \cap E_n} g d\mu = \int g d\mu_n$.

If g is a characteristic function, that is $g = \lambda_B, B \in S$. Then $\int_{A \cap E_n} g d\mu = \int_A g \lambda_{E_n} d\mu = \mu(A \cap B \cap E_n) = \mu_n(A \cap B) = \int_A \lambda_B d\mu_n = \int_A g d\mu_n$.

It follows from the homogeneity and additivity of integral that the equality holds whenever g is a non-negative simple function. Hence $\int_{A \cap E_n} g d\mu = \int_A g \mu_n$ holds for any non-negative measurable function g.

Then by Theorem 5 and $\int_{A \cap E_n} g d\mu = \int_A g d\mu, L({}^*\upsilon_n)(B) = \int_B {}^0({}^*g) d(L({}^*\mu_n))$ for each $B \in L({}^*S, {}^*\mu)$, the definition of $L({}^*\upsilon)$ and Theorem 2 imply that

$$L({}^*\upsilon(B)) = \int {}^0({}^*g) d(L({}^*\mu)).$$

We can easily obtain the following corollaries from theorem 5 and its extension.

Corollary 1 *Let μ and υ be σ-finite measures on (X, S). Then $\upsilon \ll \mu$ if and only if $L({}^*\upsilon) \ll L({}^*\mu)$ and $d(L({}^*\upsilon))/d(L({}^*\mu)) = {}^0({}^*(d\upsilon/d\mu)) d\upsilon/d\mu$ is the Radon-Nikodry derivative of υ with respect to μ).*

Corollary 2 *Let μ and υ be σ-finite measures on (X, S). If $g = d\mu/d\upsilon$, then for every measurable function $f: {}^*X \to R$, f is $L({}^*\upsilon)$-integrable if and only if $f\,0\,{}^0({}^*g)$-integrable and $\int f d(L({}^*\upsilon)) = \int f\,0\,{}^0({}^*g) d(L({}^*\mu))$.*

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Loeb 空间中的 Radon-Nikodym 导数

陈 东 立

(西安建筑科技大学数学教研室, 710055)

摘 要

设 υ 及 μ 为定义在可测空间 (X, S) 上的有限测度. 本文首先证明了若 $\upsilon \ll \mu$ (即 υ 关于 μ 绝对连续), 则有 $L({}^*S, {}^*\mu) \subset L({}^*S, \upsilon)$. 进而证明了 $\upsilon \ll \mu$ 当且仅当 $L({}^*\upsilon) \ll L({}^*\mu)$ 并且 $d(L({}^*\upsilon))/d(L({}^*\mu)) = {}^0({}^*(d\upsilon/d\mu))$ 即 Loeb 空间中的 Radon-Nikodym 定理. 本文按一种自然的方式定义了 σ-有限测度空间的 Loeb 空间, 则以上结论可以推广到 σ-有限的情况.

本文在扩大或 ω-饱和的非标准模型中讨论.

Journal of Mathematical Research & Exposition
Vol. 17, No. 2, 193- 196, May 1997

Radon-Nikodym Derivative on Loeb Space*

Chen Dongli
(Xi'an Univeristy of Architecture and Technology, 710055)

Abstract In this paper, we first show that if $\ll \upsilon$ is absolutly continuous with respect to μ, i.e., $\upsilon \ll u$, then $L({}^*S, {}^*\mu) \subset L({}^*S, {}^*\upsilon)$. We also prove that $\upsilon \ll \mu$ if and only if $L({}^*\upsilon) \ll L({}^*\mu)$ and $d(L({}^*\upsilon))/d(L({}^*\mu)) = {}^0({}^*(d\mu/d\upsilon))$. We shall define the Loeb space of σ-finite measure space by a natural way and prove that the results above can be extended to σ-finite measure spaces.

Keywords absolute continuity, Loeb space, Radon-Nikodym Theorem, Radon-Nikodym derivative.

Classification AMS(1991) 28E05/CCL O174.12

1. Loeb Spaces

Let (X, S, μ) be a totally finite measure space, then $({}^*X, {}^*S, {}^*\mu)$ is an internal, finitly additive measure space by transfer principle. The Loeb space with respect to (X, S, μ) can be defined as follows: Let $\overline{{}^*\mu}$ and $\underline{{}^*\mu}$ be maps from $P({}^*X)$ to $R^+ \cup \{0\}$ such that $\overline{{}^*\mu}(A) = inf\{{}^0({}^*\mu)(B): B \in {}^*S$ and $A \subset B\}$ and $\underline{{}^*\mu}(A) = sup\{{}^0({}^*\mu)(C): C \in {}^*S$ and $C \subset A\}$ for each subset A of ${}^*X$.

Define $L({}^*S, {}^*\mu) = \{A \subset {}^*X: {}^0\overline{\mu}(A) = {}^*\underline{\mu}(A)\}$ and $L({}^*\mu): L({}^*S, {}^*\mu) \to R \cup \{0\}$, such that $L({}^*\mu)(A) = \overline{{}^*\mu}(A) = \underline{{}^*\mu}(A)$ for each A that belongs to $L({}^*S, {}^*\mu)$. It is known that $({}^*X, L({}^*S, {}^*\mu), L({}^*\mu))$ is a complete standard measure space which is called the Loeb space with respect to measure space (X, S, μ).

If (X, S, μ) is a σ-finite measure space, we can define the Loeb space with respect to (X, S, μ) in the following natrual way: Let $\{E_n\}_{n\in N}$ be an increasing sequence of sets which terms are in S such that $X = \cup E_n$ and $\mu(E_n) < +\infty$. For each n belongs to N, define a finite measure μ_n on (X, S) by letting $\mu_n(A) = \mu(A \cap E_n)$ for each A belongs to S.

Let $L({}^*S, {}^*\mu) = \cap L({}^*S, {}^*\mu_n)$ and $L({}^*\mu): L({}^*S, {}^*\mu) \to R^+ \cup \{0\}$, such that $L({}^*\mu)(A) = \lim L({}^*\mu_n)(A)$ for each $A \in L({}^*S, {}^*\mu)$. It is obvious that $({}^*X, L({}^*S, {}^*\mu), L({}^*\mu))$ is a complete measure space. In fact $\sigma({}^*S) \subset L({}^*S, {}^*\mu)$. We call $({}^*X, L({}^*S, {}^*\mu), L({}^*\mu))$. the Loeb space with repect to the σ-finite measure space (X, S, μ).

Theorem 1 *Let* (X, S, μ) *be a finite measure space and* f *be a* μ-*integrable function, then* ${}^0({}^*f)$ *is* $L({}^*\mu)$-*integrable and* ${}^0\int {}^*f d{}^*\mu = \int {}^0({}^*f) d(L({}^*\mu))$.

Proof Without loss of generality, assume that f is a non-negative function.

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* Received Sep. 6, 1994.

Since f is μ-integrable, we have $\mu(X(f=+\infty))=0$. Then $X(f=+\infty)=\bigcap X(f\geq n)$ and $\mu(X)<+\infty$ implies that $\lim\mu(f\geq n)=\mu(X(f=+\infty))=0$. Hence for every positive ε there exists $n_0\in N$ such that $\int_{(f\geq n)} f\,d\mu<\varepsilon$ for each $n\in N$ which satisfied $n\geq n_0$ by the absolute continuty of integral. It follows from the transfer principle that $\int^*_{(f\geq n)} {}^*f\,d\,{}^*\mu<\varepsilon$.

Let H be an arbitrary infinite natrual number. Then ${}^*X({}^*f\geq H)\subset {}^*X({}^*f\geq n)$ for each $n\in N$ implies that $\int_{({}^*f\geq H)} {}^*f\,d\,{}^*\mu\approx 0$ that is, ${}^*f$ is S-integrable. Applying Theorem 3.9 in [1] to ${}^*f$, we have that ${}^0({}^*f)$ is $L({}^*\mu)$-integrable and ${}^0\int {}^*f\,d\,{}^*\mu=\int {}^0({}^*f)\,d(L({}^*\mu))$.

Theorem 2 *Let (X,S,μ) be a σ-finite measure space and f be an $L({}^*\mu)$-measurable function from ${}^*X$ to R. Then f is $L({}^*\mu)$-integrable if and only if f is $L({}^*\mu_n)$-integrable and $\sup\{\int|f|\,d(L({}^*\mu_n))\}<+\infty$. If f is $L({}^*\mu)$-integrable then $\int f\,d(L({}^*\mu))=\lim\int f\,d(L({}^*\mu_n))$.*

Proof Let f be $L({}^*\mu_n)$-integrable and $\sup\int|f|\,d(L({}^*\mu_n))<+\infty$. If f is a characteristic function, the proof is trivial. It follows from the homogeneity and additivity of integral that the sufficence is true for non-negative simple funcitons.

Without loss of generality, suppose that f is non-negative. Then there is an increasing non-negative sequence $\{\varphi_k\}_{k\in N}$ of simple functions such that $\lim\varphi_k(x)=f(x)$ for each x in ${}^*X$.

Since $\lim\int\varphi_k(x)\,d(L({}^*\mu_n))=\int\varphi_k\,d(L({}^*\mu))$ for each $k\in N$, then $\lim\lim\int\varphi_k\,d(L({}^*\mu_n))=\lim\int\varphi_k\,d(L({}^*\mu))=\int f\,d(L({}^*\mu))$.

It is not difficult to verify that

$$\lim\lim\int\varphi_k\,d(L({}^*\mu_n))=\lim\lim\int\varphi_k\,d(L({}^*\mu_n))=\lim\int f\,d(L\ {}^*\mu_n)).$$

Hence $\lim\int f\,d(L({}^*\mu))=\int f\,d(L({}^*\mu))$. Similarly, we can prove the necessarity.

2 Absolute Continuity and Radon-Nikodym Theorem

Let μ and υ be measures on a measurable space (X,S). Then υ is absolutly continuous with respect to μ if each set A that belongs to S and satisfies $\mu(A)=0$ also satisfies $\upsilon(A)=0$, which is denoted by $\upsilon\ll\mu$.

Theorem 3 *Let μ and υ be finite measures on the measureable space (X,S). Then $\upsilon\ll\mu$ if and only if for each set A that belongs to ${}^*S$ and satisfies $\mu^*(A)\approx 0$ also satisfies ${}^*\upsilon(A)\approx 0$.*

Proof First suppose $\upsilon\ll\mu$. Then for each positive ε there is a positive δ such that each S-measurable set A that satisfies $\mu(A)<\delta$ also satisfies $\upsilon(A)<\varepsilon$ (by [2], Lemma 4.2.1).

Using transfer principle, ${}^*\mu(A)<\delta$ implies ${}^*\upsilon(A)<\varepsilon$ for each A that belongs to ${}^*S$. Hence ${}^*\mu(A)\approx 0$ implies ${}^*\upsilon(A)<\varepsilon$ for each positive ε, that is, ${}^*\upsilon(A)\approx 0$.

Next suppose that ${}^*\mu(A)\approx 0$ implies ${}^*\upsilon(A)\approx 0$ for each $A\in{}^*S$. Then the following assertion is true: For each positive ε, there exists $\delta\in{}^*R$ such that $A\in{}^*S$ and ${}^*\mu(A)<\delta$ implies ${}^*\upsilon(A)<\varepsilon$ (Take δ to be infinitesimal). It follows from transfer principle that there exists $\delta\in R^+$ such that $A\in S$ and $\mu(A)<\delta$ implies $\upsilon(A)<\varepsilon$, that is $\upsilon\ll\mu$.

We have to use the following basic nonstandard tools to prove Theorem 4.

Denumerable comprehension: For every internal set A and every function $f;N\to A$, there is an internal function g; ${}^*N \to A$ extending f.

Overflow principle: Let A be internal, $A \subset {}^*N$. If $n \in A$ for all finite $n \geq n_0$, then there is an infinite $H \in {}^*N$ with $n \in A$ for all $n_0 \leq n \leq H$.

Theorem 4 *Let μ and υ be finite measures on the measurable space (X, S). If $\upsilon \ll \mu$, then $L({}^*S, {}^*\mu) \subset L({}^*S, {}^*\upsilon)$.*

Proof Let A belong to $L({}^*S, {}^*\mu)$. Because $L({}^*S, {}^*\mu)$ is the $L({}^*\mu)$-completion of $\sigma({}^*S)$, there exists $B \in \sigma({}^*S)$ and $C \in \sigma({}^*S)$ such that $B \subset A \subset C$ and $L({}^*\mu)(B - C) = 0$.

By the definition of $L({}^*\upsilon)$, there is an increasing sequence $\{P_n\}_{n\in N}$ of sets whose terms in ${}^*S$ with $|{}^*\upsilon(P_n) - L({}^*\upsilon)(B - C)| \leq 1/n$. Using denumerable comprehension and Overflow, there is an infinite $H_1 \in {}^*N$ such that $|{}^*\upsilon(P_n) - L({}^*\upsilon)(B - C)| \leq 1/n$ and $P_n \subset P_{H_1}$ for each $n \leq H_1$. Then for each $H \in {}^*N - N$, $H \leq H_1$ implies that ${}^*\upsilon(P_H) \approx L({}^*\upsilon)(B - C)$.

Since $P_n \subset B - C$ for each $n \in N$, then ${}^*\mu(P_n) \approx 0$ for each $n \in N$. By infinitesimal prolongation theorem, there exists an infinite H_2 such that ${}^*\mu(P_m) \approx 0$ for all $m \leq H_2$.

Let $H = \min\{H_1, H_2\}$, then ${}^*\mu(P_H) \approx 0$ and ${}^*\upsilon(P_H) \approx L({}^*\upsilon)(B - C)$. Since υ is absolutly continuous with respect to μ, it follows from Theorem 3 that ${}^*\upsilon(P_H) \approx 0$, that is $L({}^*\upsilon)(B - C) = 0$, hence $A \in L({}^*S, {}^*\upsilon)$.

The following theorem is the main result of this paper. We call it the Radon-Nikdoym theorem in Loeb space.

Theorem 5 *Let υ and μ be finite measures on the measurable space* (X, S) *and $\upsilon \ll \mu$ Then there exists a non-negative* S*-measurable function* g *such that*

$$L({}^*\upsilon)(B) = \int {}^0({}^*g)\, d(L({}^*\mu))$$

for each $B \in L({}^*S, {}^*\mu)$.

Proof Because $\upsilon \ll \mu$, it follows from Radon-Nikoym theorem that there exists a non-negative S-measurable function g such that $\upsilon(A) = \int g\, d\mu$ holds for each A that belongs to S. Transfer principle implies ${}^*\upsilon(A) = \int {}^*g\, d\, {}^*\mu$ for each A belongs to ${}^*S$.

Take an arbitrary set B in $L({}^*S, {}^*\mu)$. By Theorem 4, B belongs to $L({}^*S, {}^*\upsilon)$. It follows from the definition of $L({}^*\upsilon)(B)$ that $L({}^*\upsilon)(B) = \sup\{{}^0({}^*\upsilon)(A) : A \in {}^*S, A \subset B\} = \sup\{\int_A {}^0({}^*g)\, d\,{}^*\mu : A \in {}^*S, A \subset B\} = \sup\{\int_A {}^0({}^*g)\, d(L({}^*\mu)) : A \in {}^*S, A \subset B\}$.

On the other hand, by the definition of $L({}^*\mu)(B)$ there exists an increasing sequence $\{B_n\}_{n\in N}$ of sets in ${}^*S$, such that $B_n \subset B$ and $\lim L({}^*\mu)(B_n) = L({}^*\mu)(B)$. Then it follows from the absolute continuity of integral that for each positive ε there exists a set A in ${}^*S$, such that $A \subset B$ and $\int_B {}^0({}^*g)\, d(L({}^*\mu)) \leq \int_A {}^0({}^*g)\, d(L({}^*\mu)) + \varepsilon$ Hence $L({}^*\upsilon)(B) = \sup\{\int_A {}^0({}^*g)\, d(L({}^*\mu)) : A \in {}^*S, A \subset B\} = \int_B {}^0({}^*g)\, d(L({}^*\mu))$. □

Theorem 5 can be extended to the case in which μ and υ are σ-finite measures on (X, S).

Let υ and μ be σ-finite measures on the measurable space (X, S) with $\upsilon \ll \mu$ and $\{E_n\}_{n\in N}$ be an increasing sequence of sets whose terms in S, such that $X = \cup E_n$ and $\mu(E_n) < +\infty$, $\upsilon(E_n) < +\infty$ for each $n \in N$. It is clear that $\upsilon \ll \mu$ implies $\upsilon_n \ll \mu_n$ for each $n \in N$.