Journal of Mathematical Research & Exposition
Vol. 17, No. 2, 203-204, May 1997

On a Class of Fractal Interpolation Functions*

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Keywords iterated function systems, fractal interpolation functions, moments

Classification AMS(1991) 41A, 65D/CCL O174.41

Let X be a complete metric space, and let $W_i: X\to X$ for $i=1,\cdots,n$ be continuous mappings. The $\{X; W_1,\cdots,W_n\}$ is called an iterated function system, abbreviated IFS. A nonempty compact set $A\subset X$ is called an attractor of the IFS if A satisfies the set equation $A=\bigcup_{i=1}^n W_i(A)$. IFSs with their attractors provides a convenient framework for the description, classification, and communication of fractals.

In 1986, Barnsley [1] introduced a class of IFSs, each of which has a unique attractor which is the graph of some continuous function defined on a compact interval. Let $\{(x_i,y_i)\in\mathbf{R}^2; i=0,1,\cdots,n\}$ be a given set of data points with $x_0<x_1<\cdots<x_n$. We denote $I=[x_0,x_n]$ and $I_i=[x_{i-1},x_i]$ for $i=1,\cdots,n$. Let $L_i: I\to I_i$ for $i=1,\cdots,n$ be contraction homeomorphisms with $L_i(x_0)=x_{i-1}$ and $L_i(x_n)=x_i$. Let $F_i: I\times\mathbf{R}\to\mathbf{R}$ for $i=1,\cdots,n$ be continuous mappings with $F_i(x_0,y_0)=y_{i-1}$, $F_i(x_n,y_n)=y_i$ and, for some constant $q\in[0,1)$, $|F_i(u,v_1)-F_i(u,v_2)|\le q|v_1-v_2|$ for any $u\in I$, v_1 and $v_2\in\mathbf{R}$. Now define mappings $W_i: I\times\mathbf{R}\to I\times\mathbf{R}$ by $W_i(x,y)=(L_i(x),F_i(x,y))$ for $i=1,\cdots,n$. Then for some compact interval $[a,b]$, the IFS$\{I\times[a,b]; W_1,\cdots,W_n\}$ has a unique attractor G. G is the graph of a continuous function $f: I\to[a,b]$ which satisfies

$$f(x_i)=y_i,\qquad i=0,1,\cdots,n.$$

Such an f is called, by Barnsley, a fractal interpolation function, abbreviated FIF, for the fractal dimension of G is usually fractional.

In this note we generalize Barnsley's construction of FIFs and answer a question of Barnsley and Harrington[2] in a more extended sense.

Let $x_0<x_1<\cdots<x_n$ and let $y_0,y_1,\cdots,y_n\in\mathbf{R}$. *We denote* $I=[x_0,x_n]$, $J=\{1,\cdots,n\}$ *and* $I_i=[x_{i-1},x_i]$ *for* $i\in J$. *Let* J_1 *and* J_2 *be two subsets of* J *with* $J_1\cup J_2=J$ *and* $J_1\cap J_2=\emptyset$. *Let* $L_i: I\to I_i$ *for* $i\in J$ *be contraction homeomorphism satisfying*

* Received Jun. 29, 1994.

$$L_i(x_0) = \begin{cases} x_{i-1}, & \text{if } i \in J_1 \\ x_i, & \text{if } i \in J_2 \end{cases} \quad (1)$$

$$L_i(x_n) = \begin{cases} x_i, & \text{if } i \in J_1 \\ x_{i-1}, & \text{if } i \in J_2 \end{cases} \quad (2)$$

Let $F_i: I \times \mathbf{R} \to \mathbf{R}$ for $i \in J$ be continuous mappings with

$$F_i(x_0, y_0) = \begin{cases} y_{i-1} & \text{if } i \in J_1 \\ , y_i, & \text{if } i \in J_2 \end{cases} \quad (3)$$

$$F_i(x_n, y_n) = \begin{cases} y_i, & \text{if } i \in J_1 \\ y_{i-1}, & \text{if } i \in J_2 \end{cases} \quad (4)$$

and, for some constant $\alpha \in [0, 1)$,

$$|F_i(u, v_1) - F_i(u, v_2)| \leq \alpha |v_1 - v_2|$$

for all $u \in I$, $v_i \in \mathbf{R}$, $v_2 \in \mathbf{R}$ and $i \in J$.

Now define $\varphi_i \in C(I)$ by $\varphi_i(x) = F_i(x, y_0)$ for $i \in J$. Denote $h = \max\{|y_0|, \|\varphi_1\|_\infty, \cdots, \|\varphi_m\|_\infty\}$, $\eta = (h + |y_0|)/(1 - \alpha)$, $a = y_0 - \eta$, $b = y_0 + \eta$ and $K = I \times [a, b]$. Then we have the following condusion.

Theorem 1 *The IFS $\{K; W_i, i \in J\}$ defined as above has a unique attractor G, G is the graph of a continuous function $f: I \to [a, b]$ which obeys*

$$f(x_i) = y_i, \ i = 0, 1, \cdots, n.$$

Now we further assume that $L_i: I \to I_i$ for $i \in J$ are affine and $F_i: I \times \mathbf{R} \to \mathbf{R}$ for $i \in J$ have the following forms,

$$F_i(x, y) = \alpha_i y + q_i(x),$$

where $\alpha_i \in (-1, 1)$, $|\alpha_i| \leq \alpha$, $q_i \in C(I)$ and $i \in J$.

Theorem 2 *Let the IFS $\{K; W_i, i \in J\}$ defined as in Theorem 2 satisfy the above assumption. Let f be an FIF generated by the IFS. For each $m \in \{0, 1, 2, \cdots\}$ one can evaluate explicitly the moment integral $f_m = \int_I x^m f(x) dx$ in terms of $f_{m-1}, f_{m-2}, \cdots, f_0$, the interpolation points $\{x_0, x_1, \cdots, x_n\}$, the scaling parameters $\{\alpha_i: i \in J\}$ and the moments $Q_{k,i} = \int_I x^k q_i(x) dx$ where $k \in \{0, 1, \cdots, m\}$ and $i \in J$.*

References

[1] M. F. Barnsley, *Fractal functions and interpolation*, Constr. Approx., 2(1986), 303-329.

[2] M. F. Barnsley and A. N. Harrington, *The calculus of fractal interpolation functions*, J. Approx. Theory., 57(1989), 14-35.