Journal of Mathematical Research & Exposition
Vol. 17, No. 2, 205-206, May 1997

# Riordan-Lagrange Inverse Relation*

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**Keywords** Riordan group, inverse relation, Lagrange inverse formula.

**Classification** AMS(1991) 05A15, 05A19/CCL O157.1

In 1991, L. V. Shapiro et al introducted the concept of Riordan group [1]. Recently, R. Sprugnoli has studied the theory thoughly and gave new proofs of many combinatorial identities systematically [2, 3]. The main point of the theory is to establish the connection between an array and a couple of generating functions. The generating function can then be used to obtain the closed form of a combinatorial sum or its asymptotic value. However, the importance of the connection between Riordan array and inverse relation has been underestimated. As a result, our aim is to develop a new inverse relation.

Let $\mathbf{F} = \mathbf{R}[t]$ be a ring of formal power series, and $g(t), f(t) \in \mathbf{F}$, i.e., $g(t) = \sum_k g_k t^k$, $f(t) = \sum_k f_k t^k$ and $f_0 = 0$. The sequence of functions $\{d_k(t)\}_{k\in N}$ iteratively defined by

$$\begin{cases} d_0(t) = g(t) \\ d_k(t) = g(t)[f(t)]^k, \end{cases} \tag{1}$$

defines an infinite triangle $\{d_{n,k} | k, n \in N, k \le n\}$ in which $d_{n,k} = [t^n]d_k(t)$, where $[t^n]d_k(t)$ means the coefficient of $t^n$ in expansion of $d_k(t)$ in $t$. The $\{d_{n,k}\}$ is called a Riordan array of $d_k(t)$ in $t$ and write $\{d_{n,k}\} = (g, f)$.

Let $A = \{a_k\}, B = \{b_k\}$ be sequences, $D = \{d_{n,k}\} = (g, f)$ be a Riordan array. If $A = DB$, i.e., $a_n = \sum_{k=0}^{\infty} d_{n,k} b_k$, $n = 0, 1, 2, \cdots$. By the properties of Riordan group, $D^{-1}$ exists and $D^{-1} = (1/g(\bar{f}), \bar{f})$, where $f(\bar{f}(x)) = \bar{f}(f(x)) = x$.

**Example** Given $a_n = \sum_k \binom{n}{k} b_k$, we can easily find $D = (\frac{1}{1-x}, \frac{x}{1-x})$. Obviously $\bar{f}(x) = \frac{x}{1+x}$, $D^{-1} = (\frac{1}{1+x}, \frac{x}{1+x})$, then

$$\bar{d}_{n,k} = [x^n]\frac{1}{1+x}\left(\frac{x}{1+x}\right)^k = [x^{n-k}](1+x)^{-(k+1)} = (-1)^{n+k}\binom{n}{k}$$

Therefore $b_n = \sum (-1)^{n+k}\binom{n}{k} a_k$ as we known.

* Received Feb. 24, 1996. The research is suuported by NNSF.

For arbitrary function $f(x)$, it is not easy to find its reciprocal function $\bar{f}(x)$. By using Lagrange Inversion Formula we have the following

**Theorem** *Let $D=\{d_{n,k}\}=(g,f)$ be a Riordan array, then the following inverse relation holds:*

$$\begin{cases} a_n = \sum_k d_{n,k} b_k \\ b_n = \sum_k \frac{1}{n}[t^{n-1}]\{[(g^{-1}(t))' t^k + k(g^{-1}(t))t^{k-1}](\frac{f(t)}{t})^{-n}\}a_k \end{cases} \quad (2)$$

where $n=0,1,2,\cdots$.

The Theorem is a generator of inverse relations If a couple of functions $f, g$ are given, then we can get a inverse relation through (2), and the inverse relation includes almost all inverse relations in the book "Combinatorial Identities" by J. Riordan.

**Example** Legendre-Chebyshev type inverse relation

$$\begin{cases} a_n = \sum_k (-1)^{n+1}\binom{n+p+(c-1)k}{n-k} b_k \\ b_n = \sum_k \frac{p+ck+1}{(c-1)n+p+k+1}\binom{cn+p}{n-k} a_k \end{cases}$$

It may be verified that $D=(d_{n,k})=((1+t)^{-(p+1)}, t(1+t)^{-c})$. Thus the Theorem implies

$$\begin{aligned} b_n &= \sum_k \frac{1}{n}[t^{n-1}][(1+t)^{p+1}]' t^k + k(1+t)^{p+1}t^{k-1}](1-t)^{cn}a_k \\ &= \sum_k \frac{1}{n}[t^{n-1}]\{(p+1)t^k(1+t)^{cn+p} + kt^{k-1}(1+t)^{cn+p+1}\}a_k \\ &= \sum_k \frac{1}{n}\left\{(p+1)\binom{cn+p}{n-k-1} + k\binom{cn+p+1}{n-k}\right\}a_k \\ &= \sum_k \frac{p+ck+1}{(c-1)n+p+k+1}\binom{cn+p}{n-k}a_k \end{aligned}$$

The Riordan-Lagrange Inverse Relation can also be obtained equivalently from Generalized Stirling Number Pairs

## Refereces


[1] L. V. Shapiro, S. Getu, W-J. Woan and Woodson, *The Riordan group*, Discrete Appl. Math., **34** (1991), 229-239.

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[3] R. Sprugnoli, *Riordan arrays and Abel-Gould identity*, Discrete Math., **142** (1995), 213-233.

[4] L. C. Hsu, *Theory and Application of Generalized Stirling Number Pairs*, J. of Math R. and E. **9**: **2** (1989).