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Conformally Flat Minimal Hypersurfaces in a Hyperbolic Space*

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Abstract In this paper it is proved that all the conformally flat minimal hypersurfaces in a hyperbolic space are either rotational hypersurfaces or glued by some such pieces via totally geodesic ones. Combining this result with a previous theorem of Wang Xinmin and Xu Zhicai, We generalize a theorem of Blair about a generalization of catenoid.

Key words minimal submanifolds, conformal flatness, hyperbolic space.

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1. Introduction and main result

As a classical theorem, it is well known that the catenoid is the only minimal surface of revolution in the three dimensional Euclidean space E^3. Of course it is conformally flat. Professor David E. Blair [1] had obtained an interesting generalization of this property as follows:

Theorem A *Let M^n, $n \geq 4$, be a conformally flat, minimal hypersurface immersed in E^{n+1}. Then M^n is either a hypersurface of revolution $S^{n-1} \times M^1$ where S^{n-1} is a Euclidean sphere and M^1 is a plane curve whose curvature κ as a function of arc length. s is given by $\kappa = -(n-1)\alpha$, $\alpha = -1/v$ and*

$$s = \int \frac{v^{n-1} dv}{|Av^{2n-1} - 1|^{1/2}},$$

where A is a constant, or M^n is totally geodesic.

Recently Professor Wang Xinmin and Professor Xu Zhicai[2] determined all minimal hypersurfaces of revolution in a hyperbolic space. They proved that the rotational minimal hypersurfaces in the hyperbolic space must be hyperplanes or generalized catenoids.

Comparing the results in [1] and [2], one can ask: whether all the conformally flat minimal hypersurfaces in a hyperbolic space are of revolution of not?

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The aim of this paper is to give an answer to this question. We shall prove the following

Theorem *Let M^n be a conformally flat, minimal hypersurface immersed in an $(n+1)$-dimensional hyperbolic space H^{n+1}, $n\geq 4$. Then either M^n is a hypersurface of revolution $\Sigma^{n-1}\times M^1$, where Σ^{n-1} is either a geodesic sphere, a horosphere, or an equidistant hypersurface, and M^1 is a plane curve, i.e., a curve which lies in a totally geodesic surface H^2, or M^n is glued by some such pieces via totally geodesic ones.*

Combining this Theorem with the Theorem 1 of Wang and Xu[2], we may generalize the result of Blair[1] to the case where the ambient manifold is a hyperbolic space.

The manifolds considered here are assumed to be smooth and connected.

2 Proof of Theorem

Let H^{n+1} be an $(n+1)$-dimensional hyperbolic space of negatively constant curvature K_0, $n\geq 4$. Of course it is conformally flat. According to a theorem of Cartan and Schouten, every conformally flat hypersurface of a conformally flat space of dimension great than 4 is quasiumbilical (see, for example, Chen[3], p. 154). Hence the conformally flat hypersurface M^n of H^{n+1} is quasiumbilical, i.e., there exist functions α and β and a unit tangent vector field U with its dual 1-form ω, such that

$$h = \alpha g + \beta\omega\otimes\omega,$$

where g and h are the metric and the second fundamental tensors, respectively.

Now in H^{n+1} we choose a local section of the orthonormal frame bundle $\{E_\mu\}$, $\mu=0, 1, 2, \cdots, n$, such that E_0 is normal to M^n, $E_1=U$ (so it is tangent to M^n), $E_2, \cdots, E_n$ are tangent to M^n and orthogonal to U. In this frame the tensor h has components as follows:

$$h_{11}\equiv h(U,U) = \alpha+\beta,\quad h_{jj}\equiv h(E_j,E_j) = \alpha,$$
$$h_{1j}\equiv h(U,E_j) = 0,\quad h_{jk}\equiv h(E_j,E_k) = 0\ (j\neq k),$$

where $j, k=2, \cdots, n$.

Moreover, since M^n is minimal, it follows that

$$h_{11} = -(n-1)\alpha.$$

Let $\{\omega^\mu\}$ be the dual basis in the cotangent bundle of H^{n+1}, the structure equations of H^{n+1} are

$$d\omega^\mu = \omega^\nu\wedge\omega^\mu_\nu,\quad d\omega_{\mu\nu} - \omega^\tau_\mu\wedge\omega_{\tau\nu} = -\frac{1}{2}\overline{R}_{\mu\nu\sigma\tau}\omega^\sigma\wedge\omega^\tau,$$

where

$$\overline{R}_{\mu\nu\sigma\tau} = K_0(g_{\mu\sigma}g_{\nu\tau} - g_{\mu\tau}g_{\nu\sigma}),\quad \omega_{\mu\nu} = \gamma_{\mu\nu\tau}\omega^\tau.$$

Restricting onto M^n, we have $\omega^0=0$, and then get the structure equations of M^n, the Gauss equations, and the Codazzi equations. Note that

$$\gamma_{a0b} = h_{ab}\,(a, b = 1, 2, \cdots, n),$$

from Codazzi equations we get

$$a_{\cdot j} = 0, \tag{1}$$

$$a\gamma_{1j1} = 0, \tag{2}$$

$$a(\gamma_{1jk} - \gamma_{1kj}) = 0 \ (j \neq k), \tag{3}$$

$$a_{\cdot 1} - na\gamma_{j1j} = 0, \tag{4}$$

where $j, k = 2, \cdots, n, j \neq k$, and

$$a_{\cdot b} = E_b(a) \ (b = 1, 2, \cdots, n).$$

If $a = 0$ at some point A, then by (1) and (4) we know $da = 0$ at A. Hence the function a does not have any isolated zero point.

If a is identically zero, then M^n is totally geodesic in H^{n+1}, hence it is a hyperplane, clearly it is a hypersurface of revolution.

If $a \neq 0$ at some point $x \in M^n$, we may consider the domain Ω containing x in which $a \neq 0$ everywhere. Then by (2), (3) and (4), we get

$$\gamma_{1j1} = 0, \ \gamma_{j1k} = \gamma_{k1j}, \ \gamma_{j1j} = \gamma_{k1k},$$

where $j, k = 2, \cdots, n; j \neq k$. Therefore

$$d\omega^1 = \omega^a \wedge \omega_a^1 = \gamma^1_{ab}\omega^a \wedge \omega^b = \gamma^1_{jk}\omega^j \wedge \omega^k + \gamma^1_{j1}\omega^j \wedge \omega^1$$

$$= \frac{1}{2}(\gamma_{j1k} - \gamma_{k1j})\omega^j \wedge \omega^k - \gamma_{1j1}\omega^j \wedge \omega^1 = 0.$$

So, in Ω, M^n can be foliated by a system of totally umbilical hypersurface Σ^{n-1}, the integral submanifolds defined by $\omega^1 = 0$. Considering both the Gauss equations for M^n in H^{n+1} and for Σ^{n-1} in M^n, it follows that such a Σ^{n-1} is a submanifold with all points umbilics in H^{n+1}, and hence it is either a geodesic sphere, a horosphere, or an equidistant surface. Therefore, in the domain for which $a \neq 0$, there exists a decomposition $M^n = \Sigma^{n-1} \times M^1$, where M^1 is an integral curve of the vector field U.

Now one computes the curvatures of the curve M^1 in H^{n+1}. Since the tangent vector field of M^1 is U, by the consequence of Codazzi equations, we find that the first curvature is

$$\kappa_1 = (n - 1)|a|,$$

and the second curvature is

$$\kappa_2 = 0,$$

therefore, by a theorem about the curves in a manifold of constant curvature (see, for example, Spivak[4]), M^1 lies in a 2-dimensional plane spanned by E_0 and U in H^{n+1}. Denote this plane by P.

If each Σ^{n-1} is a geodesic sphere, then its center lies in the plane P. Considering another integral curve of U, we obtain another plane P' instead of P. All the centers of these spherical leaves of the foliation of M^n lie in both P and P', and then in their intersection Γ, which is a geodesic of ambient manifold H^{n+1}. Let G be the isometry group of H^{n+1}, and G_0 its subgroup which keeps each point of Γ fixed. Then every geodesic sphere Σ^{n-1} is an orbit of G_0. So M^n is a hypersurface of revolution if $a\neq 0$.

If Σ^{n-1} is a horosphere, then it lies in a totally geodesic hypersurface H^n. Denote by B^{n+1} and B^n the Poicare's disk models for H^{n+1} and H^n, resdectively. Then the horosphere Σ^{n-1} is shown by an $(n-1)$-sphere tangent to the boundary of the model sphere B^n at a point A (the "infinite point" in the direction of the normal of Σ^{n-1} in H^n). The point A lies on the boundary of the model sphere B^{n+1}, too. Now in H^{n+1} the "infinite point" A lies in both plane P and P', and then in their intersection Γ. It is the same for all those leaves of type horosphere, so in this case M^n is a hypersurface of revolution, too, and the geodesic Γ is just its rotational axis.

Finally, let Σ^{n-1} be an equidistant hypersurface. Denote by x the point at which the leaf Σ^{n-1} and the curve M^1 intersect. By assume $a\neq 0$, so the first curvature of M^1 does not vanish. Hence M^1 is not a geodesic. Let Γ be the geodesic of H^{n+1} throughing x and tangent to M^1. Let Σ' be another leaf neighbouring Σ^{n-1}, and x' the intersection of Σ' and Γ. The integral curve of vector field U throughing the point x' is a plane curve M', and the corresponding plane will be denoted by P'. Two hyperbolic planes P and P' intersect in the geodesic Γ. Both the equidistant hypersurfaces Σ^{n-1} and Σ' are perpendicular to the geodesic Γ, so these two equidistant hypersurfaces have a common base hyperplane. In the same way we know that all those leaves of type equidistant hypersurfaces have a common base hyperplane, and then have a common perpendicular geodesic Γ. From this it follows that M^n is a hypersurface of revolution with geodesic Γ as its rotational axis.

Thus the Theorem is completely proved.

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双曲空间中的共形平坦极小超曲面

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摘 要

本文证明了双曲空间中的共形平坦极小超曲面必为旋转超曲面或由一些旋转超曲面片用全测地超曲面片粘合而成. 将这结果与王新民及许志才的一个已发表定理相组合, 推广了Blair 关于推广悬链面的一个定理.