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A Proof on the Convexity of Functional Splines*

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Abstract We give on alternative proof of theorem 4.4 in [Hartmann et al. '93].

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The functional splines has become one of the basic methods in CAGD. Hartmann and Feng pointed out an important result for its convexity. We attempt to prove this theorem by a more direct and more elementary method and avoid the ideas of quasiconvex and Hessian matrix etc.

Definition 1 *For $C\subset R^d$ and $f:C\to R$ we difine*

$$S(f,\alpha):=\{x\in C\,|\,f(x)=\alpha\},\ S(f,\alpha)\ \textit{is called levelsurface},$$
$$U(f,\alpha):=\{x\in C\,|\,f(x)\geq\alpha\},\ U(f,\alpha)\ \textit{is called upper-levelset},$$
$$V(f,\alpha):=\{x\in C\,|\,f(x)>\alpha\}.$$

Definition 2 *For f and g be two piecewise C^1-continous functions and $C\subset U(f,0)\cap U(g,0)$ with*

(1) $C\neq\emptyset$.

(2) $\Gamma(f,g):=S(f,0)\cap S(g,0)\neq\emptyset$.

(3) $\nabla f(x)\neq\emptyset$ *for "nearly" all points* $x\in\Gamma(f,g)$. *Then*

$$F:=(1-\mu)f-\mu g^n,\ 0<\mu<1,n\geq 2,$$

is call a functional spline related to f and g.

Definition 3 *A subset D of R^n is said to be convex if $(1-\lambda)x+\lambda y\in D$ whenever $x\in D$, $y\in D$ and $0<\lambda<1$.*

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Definition 4 *A surface in R^n is called convex if it is a continuous boundary of a convex set of R^n.*

Lemma 1 *Let surface $F(x)=0$ divided R^{d+1} into regions Ω_i, $i=1,2,\cdots,h$. Then, in every Ω_i, function $F(x)$ is positive or negative.*

Lemma 2 *The generalization of arithmetic-geometric mean inequality:*

$$\left(\sum_{i=1}^{n} p_i\alpha_i \Big/ \sum_{i=1}^{n} p_i\right)^{\sum_{i=1}^{n} p_i} \geq \prod_{i=1}^{n} \alpha_i^{p_i},$$

where $p_i, \alpha_i \in R^+$. The strict inequality holds iff all the α_i's are equal.

Lemma 3 *Let $n_i=(a_{i1}, a_{i2}, \cdots, a_{id}) \in R^d$, $f_i(x)=n_i x+c_i$, $i=1,\cdots,m$;*

$$span\{n_1, n_2, \cdots, n_m\} = R^d.$$

Then there exists $i_0 \in \{1,2,\cdots,m\}$, such that $f_{i_0}(x^1) \neq f_{i_0}(x^2)$, $(x^1 \neq x^2)$.

Theorem *Let $n_i=(a_{i1}, a_{i2}, \cdots, a_{id}) \in R^d$, $f_i(x)=n_i x+c_i$, where $a_{ik}, c_i \in R$, $\lambda_i \in R^+$ for $i=0,1,\cdots,m$; $k=1,2,\cdots,d$.*

$$C := \bigcap_{i=1}^{m} V(f_i, 0) \text{ where } V(f_i, 0) = \{x \in R^d \mid f_i(x) > 0\}.$$

Then the functional spline $S(f_\mu, 0)$ with $F_\mu = (1-\mu)\prod_{i=1}^{m} f_i^{\lambda_i} - \mu f_0^{\lambda_0}$ is strictly convex if

$$\text{span}\{n_1, n_2, \cdots, n_m\} = R^d \tag{1}$$

and

$$\lambda_0 \geq \sum_{1}^{m} \lambda_i \tag{2}$$

Proof (a) We demonstrate that the functional spline surface $S(F_\mu, 0)$ contains no singular point on C.

Since the geometric property of surface is not related to coordinate system, so we can suppose that $f_0(0)=c_0$, $f_i(0)=c_i \geq 0$, $i=1,2,\cdots,m$ and at least one strict inequality holds

If there exists point $x^0=(x_1^0, x_2^0, \cdots, x_d^0)^T \in C$, such that $\nabla F_\mu(x^0)=0$, i.e.

$$\frac{\partial F_\mu}{\partial x_k} = (1-\mu)\prod_{i=1}^{m} f_i^{\lambda_i}(x^0)\left(\sum_{j=1}^{m} \frac{\lambda_j a_{jk}}{f_j(x^0)}\right) - \mu f_0^{\lambda_0}(x^0)\frac{\lambda_0 a_{0k}}{f_0(x^0)}$$

$$= (1-\mu)\prod_{i=1}^{m} f_i^{\lambda_i}(x^0)\left(\sum_{j=1}^{m} \frac{\lambda_j a_{jk}}{f_j(x^0)} - \frac{\lambda_0 a_{0k}}{f_0(x^0)}\right) = 0, \quad k=1,2,\cdots,d,$$

Then since $(1-\mu)\prod_{i=1}^{m} f_i^{\lambda_i}(x^0) \neq 0$, we have

$$\sum_{j=1}^{m} \frac{\lambda_j a_{jk}}{f_j(x^0)} - \frac{\lambda_0 a_{0k}}{f_0(x^0)} = 0, \quad k=1,2,\cdots,d.$$

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Thus

$$\sum_{k=1}^{d}\left(\sum_{j=1}^{m}\frac{\lambda_j a_{jk}x_k^0}{f_j(x^0)}-\frac{\lambda_0 a_{0k}x_k^0}{f_0(x^0)}\right)=0.$$

Hence

$$\sum_{j=1}^{m}\left(\lambda_j-\frac{\lambda_j c_j}{f_j(x^0)}\right)-\lambda_0=0.$$

Since $\sum_{j=1}^{m}\frac{\lambda_j c_j}{f_j(x^0)}>0$, we have $\lambda_0<\sum_{j=1}^{m}\lambda_j$. This is a contradiction to $\lambda_0\geq\sum_{j=1}^{m}\lambda_j$.

Therefore, $S(F_\mu,0)$ is a regular simple surface.

(b) We prove that there is no inflection point on $S(F_\mu,0)$.

For arbitary two different points $x^j\in F(F_\mu,0)$, $j=1,2$, i.e.,

$$F_\mu(x^1)=F_\mu(x^2)=0.$$

We consider point $x=(1-\lambda)x^1+\lambda x^2$, $0<\lambda<1$. If $F_\mu((1-\lambda)x^1+\lambda x^2)>0$ or $F_\mu((1-\lambda)x^1+\lambda x^2)<0$, then by Lemma 1 there is no inflection point on $S(F_\mu,0)$.

Because every f_i is an affine function, $i=1,2,\cdots,m$,

$$f_0(x^j)=\left(\frac{1-\mu}{\mu}\prod_{i=1}^{m}f_i^{\lambda_i}(x^j)\right)^{1/\lambda_0},\quad j=1,2$$

Thus

$$\begin{aligned}
&F_\mu((1-\lambda)x^1+\lambda x^2)\\
&=(1-\mu)\prod_{i=1}^{m}f_i^{\lambda_i}((1-\lambda)x^1+\lambda x^2)-\mu f_0^{\lambda_0}((1-\lambda)x^1+\lambda x^2)\\
&=(1-\mu)\prod_{i=1}^{m}((1-\lambda)f_i(x^1)+\lambda f_i(x^2))^{\lambda_i}-\mu((1-\lambda)f_0(x^1)+\lambda f_0(x^2))^{\lambda_0}\\
&=(1-\mu)\prod_{i=1}^{m}((1-\lambda)f_i(x^1)+\lambda f_i(x^2))^{\lambda_i}\\
&\quad-(1-\mu)\left((1-\lambda)\left(\prod_{i=1}^{m}f_i^{\lambda_i}(x^1)\right)^{1/\lambda_0}+\lambda\left(\prod_{i=1}^{m}f_i^{\lambda_i}(x^2)\right)^{1/\lambda_0}\right)^{\lambda_0}
\end{aligned}\tag{3}$$

By Lemma 2 and Lemma 3 we obtain

$$\begin{aligned}
\lambda_0&=\sum_{i=1}^{m}\lambda_i+\left(\lambda_0-\sum_{i=1}^{m}\lambda_i\right)\\
&=\sum_{i=1}^{m}\lambda_i\frac{(1-\lambda)f_i(x^1)}{(1-\lambda)f_i(x^1)+\lambda f_i(x^2)}+\left(\lambda_0-\sum_{i=1}^{m}\lambda_i\right)(1-\lambda)\\
&\quad+\sum_{i=1}^{m}\lambda_i\frac{\lambda f_i(x^2)}{(1-\lambda)f_i(x^1)+\lambda f_i(x^2)}+\left(\lambda_0-\sum_{i=1}^{m}\lambda_i\right)\lambda\\
&>\lambda_0\left(\prod_{i=1}^{m}\left(\frac{(1-\lambda)f_i(x^1)}{(1-\lambda)f_i(x^1)+\lambda f_i(x^2)}\right)^{\lambda_i}(1-\lambda)^{(\lambda_0-\sum_{i=1}^{m}\lambda_i)}\right)^{1/\lambda_0}
\end{aligned}$$

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$$+ \lambda_0 \left(\prod_{i=1}^{m} \left(\frac{\lambda f_i(x^2)}{(1-\lambda) f_i(x^1) + \lambda f_i(x^2)} \right)^{\lambda_i} \lambda^{(\lambda_0 - \sum_{i=1}^{m} \lambda_i)} \right)^{1/\lambda_0} \tag{4}$$

From (4), we have

$$\prod_{i=1}^{m} ((1-\lambda) f_i(x^1) + \lambda f_i(x^2))^{\lambda_i} > \left((1-\lambda) \left(\prod_{i=1}^{m} f_i^{\lambda_i}(x^1) \right)^{\frac{1}{\lambda_0}} + \lambda \left(\prod_{i=1}^{m} f_i^{\lambda_i}(x^2) \right)^{\frac{1}{\lambda_0}} \right)^{\lambda_0}.$$

Combining (a) and (b), $S(F_\mu, 0)$ is strictly convex.

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关于函数样条一个凸性定理的证明

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摘　要

本文的主要结果是给出文献[1]中函数样条凸性定理4.4的一个直接的证明

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