加权Fan Ky不等式及其加细*

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摘　要　本文简证了加权Ky Fan不等式，给出了两种加细形式

关键词　加权平均，Ky Fan不等式，严格单调
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本文中设 $x_i\in(0,\frac{1}{2}]$, $p_i>0$, $i=1,2,\cdots,n$, $\Sigma=\sum_{i=1}^{n}$, $\Pi=\prod_{i=1}^{n}$, $P_n=\sum_{i=1}^{n}p_i$. 分别以 $A_n=\frac{\sum p_ix_i}{P_n}$, $G_n=\Pi x_i^{\frac{p_i}{P_n}}$, $A_n'=\frac{\sum p_i(1-x_i)}{P_n}$, $G_n'=\Pi(1-x_i)^{\frac{p_i}{P_n}}$ 表示诸 x_i 及诸 $(1-x_i)$ 的加权算术平均和加权几何平均. 有Ky Fan不等式[1]

$$\frac{\Pi x_i}{(\Sigma x_i)^n}\leq\frac{\Pi(1-x_i)}{[\Sigma(1-x_i)]^n}\tag{1}$$

定理1

$$\frac{G_n}{G_n'}\leq\frac{A_n}{A_n'}\tag{2}$$

证明　函数 $f(x)=\ln\frac{1-x}{x}$, $x\in(0,\frac{1}{2}]$ 是连续下凸函数，由Jensen不等式 $f(\frac{1}{P_n}\sum p_ix_i)\leq\frac{1}{p_n}\sum p_if(x_i)$ 得

$$P_n\ln\left(\frac{1-\frac{1}{P_n}\sum p_ix_i}{\frac{1}{P_n}\sum p_ix_i}\right)\leq\sum p_i\ln\frac{1-x_i}{x_i}$$

即 $\left[\frac{\sum p_i(1-x_i)}{\sum p_ix_i}\right]^{P_n}\leq\Pi\left(\frac{1-x_i}{x_i}\right)^{p_i}$. □

上述证明较之用归纳法和反向归纳法等证法简易[2,3,4]. 当诸 $p_i=\frac{1}{n}$ 时，(2)式即为(1)式

与(2)中此式对应的差式是下面的不等式

定理2

$$G_n-G_n'\leq A_n-A_n'\tag{3}$$

证明　记 $f(x_1,\cdots,x_n)=G_n-G_n'-A_n+A_n'$. 设其最大值在 $\bar{a}=(a_1,\cdots,a_n)\in[0,\frac{1}{2}]^n$ 达到

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今证 $a_1 = a_2 = \cdots = a_n$.

若 $\bar{a}$ 是 $[0, \frac{1}{2}]^n$ 的内点, 有 $\nabla f(a_1, \cdots, a_n) = 0$, 计算得

$$\frac{P_n}{p_i} f'_{x_i} = \frac{G_n}{x_i} + \frac{G_n'}{1 - x_i} - 2$$

记 $p(x) = (1-x)G_n + xG_n' - 2x(1-x)$, 为 x 的二次式 $p(0) > 0$, $2p(\frac{1}{2}) = G_n + G_n' - 1 \le A_n + A_n' - 1 = 0$, 知 $P(x)$ 在 $(0, \frac{1}{2})$ 内恰有一个实根, 故得 $a_1 = a_2 = \cdots = a_n$, 于是有

$$f(x_1, \cdots, x_n) \le f(a_1, \cdots, a_n) = 0, \ \forall (x_1, \cdots, x_n) \in [0, \frac{1}{2}]^n.$$

若 $\bar{a}$ 是 $[0, \frac{1}{2}]^n$ 的边界点, 分两种情况:

(1) $\bar{a}$ 的各分量均不为0, 设有 $l(\ge 1)$ 个为 $\frac{1}{2}$, 不妨设 $a_{k+1} = \cdots = a_n = \frac{1}{2}$, $1 \le n - k = l \le n - 1$.

记 $h(x_1, \cdots, x_k) = f(x_1, \cdots, x_k, \frac{1}{2}, \cdots, \frac{1}{2}) = (\frac{1}{2})^{\frac{P_n - P_k}{P_n}} [G_k^{\frac{P_k}{P_n}} - (G_k')^{\frac{P_k}{P_n}}] + \frac{p_1(1-2x_1) + \cdots + p_k(1-2x_k)}{P_n}$, 有

$$h(a_1, \cdots, a_k) \ge h(x_1, \cdots, x_k), \quad \forall (x_1, \cdots, x_k) \in [0, \frac{1}{2}]^k, \tag{4}$$

其中 $0 < a_i < \frac{1}{2}(i = 1, 2, \cdots, k)$, 有 $\nabla h(a_1, \cdots, a_k) = 0$. 计算可得

$$\frac{P_n}{p_i} h'_{x_i} = (\frac{1}{2})^{\frac{P_n - P_k}{P_k}} [G_k^{\frac{P_k}{P_n}} (\frac{1}{x_i}) + G_k^{\frac{P_k}{P_n}} (1 - \frac{1}{x_i})] - 2$$

令 $Q(x) = (\frac{1}{2})^{\frac{P_n - P_k}{P_k}} [G_k^{\frac{P_k}{P_n}} (1 - x) + G_k^{\frac{P_k}{P_n}} (1 - \frac{1}{x_i})] - 2$, 有 $Q(0) > 0$, $Q(\frac{1}{2}) < 0$, 知 $Q(x)$ 在 $(0, \frac{1}{2})$ 内恰有一个根, 得 $a_1 = a_2 = \cdots = a_k$. 下面证 $a_1 = \cdots = a_k = \frac{1}{2}$.

记 $\tilde{h}(x) = h(x, \cdots, x) = (\frac{1}{2})^{\frac{P_n - P_k}{P_n}} [x^{\frac{P_k}{P_n}} - (1-x)^{\frac{P_k}{P_n}}] + \frac{P_k(1-2x)}{P_n}$, $\frac{1}{\alpha}\tilde{h}'(x) = (2x)^{\alpha - 1} + [2(1-x)]^{\alpha - 1} - 2$, 其中 $\frac{P_k}{P_n} = \alpha \in (0, 1)$.

记 $P(\alpha) = (2x)^{\alpha - 1} + [2(1-x)]^{\alpha - 1} - 2$ 有 $P'(\alpha) = (2x)^{\alpha - 1}\ln(2x) + (2 - 2x)^{\alpha - 1}\ln(2 - 2x)$. $P''(\alpha) = (2x)^{\alpha - 1}\ln^2(2x) + (2-2x)^{\alpha - 1}\ln^2(2 - 2x) > 0$, $P'(\alpha)$ 严格递增, $P'(\alpha) < P'(1) = \ln(2x) + \ln(2 - 2x) = \ln 4x(1-x)$.

记 $q(x) = \ln 4x(1-x)$, $q'(x) = \frac{1-2x}{x(1-x)}$, $q''(x) = \frac{-2(x^2 - x + \frac{1}{2})}{x^2(1-x)^2} < 0$, 知 $q'(x)$ 严格递减, $q'(\frac{1}{2}) = 0$, 得 $q'(x) > 0$, 于是 $q(x)$ 严格递增, 又 $q(\frac{1}{2}) = 0$, 故 $q(x) < 0$, 即 $P'(1) < 0$, $P'(\alpha) < 0$ 知 $P(\alpha)$ 严格递减, 则 $P(\alpha) > P(1) = 0$, 即 $\tilde{h}'(x) > 0$, $\tilde{h}(x)$ 严格递增. 于是有 $h(a_1, \cdots, a_k) = \tilde{h}(a_1) < \tilde{h}(\frac{1}{2}) = h(\frac{1}{2}, \cdots, \frac{1}{2})$, 此与(4)矛盾, 由此得 $a_1 = \cdots = a_k = \frac{1}{2}$. 得证

(2) $\underline{a}$的 $l(\geq 1)$ 个分量为0, 不妨设 $a_{k+1} = \cdots = a_n = 0$, $1 \leq n - k = l \leq n - 1$. 定义 $\varphi: [0, \frac{1}{2}]^k \to R$.

$$\varphi(x_1, \cdots, x_k) = f(x_1, \cdots, x_k, 0, \cdots, 0) = -(G_k')^{\frac{P_k}{P_n}} - \frac{P_k}{P_n} A_k + \frac{P_k A_k'}{P_n} + \frac{P_n - P_k}{P_n}$$

计算得$\frac{P_n}{p_i}\varphi_{x_i}' = \frac{(G_k')^{\alpha}}{1 - x_i} - 2$, $\alpha \in (0, 1)$. 记 $\psi(\alpha) = \frac{(G_k')^{\alpha}}{1 - x_i} - 2$, $\psi''(\alpha) > 0$, $\psi(\alpha)$ 为严格下凸, $\psi(0) \leq 0$, $\psi(1) \leq 0$, 故 $\psi(\alpha) < 0$, 即 $\varphi_{x_i}' < 0$, φ严格递减, 于是

$$\varphi(x_1, \cdots, x_k) \leq \varphi(0, \cdots, 0) = 0, \quad \forall (x_1, \cdots, x_k) \in [0, \frac{1}{2}]^k$$

由此得知 $a_1 = a_2 = \cdots = a_k = 0$. □

不等式(2)的一个等价形式是

$$\frac{A_n'}{G_n'} \leq \frac{A_n}{G_n}. \tag{5}$$

现证明(5)的两个加细形式

定理3 (加细一)

$$\frac{A_n'}{G_n'} \leq \frac{1 - G_n'}{1 - A_n'} \leq \frac{A_n}{G_n}. \tag{6}$$

证明 函数 $f(x) = x(1 - x)$ 在$[\frac{1}{2}, +\infty)$ 上严格递减, 又$\frac{1}{2} \leq G_n' \leq A_n' < 1$, 故 $f(A_n') \leq f(G_n')$. 得(6) 式左边成立

由$A_n + A_n' = 1$ 并且(3) 式得

$$G_n(1 - G_n') \leq G_n(A_n + A_n' - G_n') \leq G_n(2A_n - G_n) \leq A_n^2$$

得(6)式右边成立 □

定理4 (加细二)

$$\frac{A_n'}{G_n'} \leq \frac{1 - G_n}{1 - A_n} \leq \frac{A_n}{G_n} \tag{7}$$

证明 $f(x) = x(1 - x)$在$(0, \frac{1}{2}]$上严格递增, $0 < G_n \leq A_n \leq \frac{1}{2}$, 可得(7)式右边成立

今证左边不等式(与定理2证明过程类似).

证明 $g(x_1, \cdots, x_n) = (1 - G_n)G_n' - (1 - A_n)^2$, 设其最小值在 $\underline{a} = (a_1, \cdots, a_n) \in [0, \frac{1}{2}]^n$ 达到. 今证 $a_1 = \cdots = a_n$.

若 $\underline{a}$是$[0, \frac{1}{2}]^n$ 的内点, 有$\nabla g(a_1, \cdots, a_n) = 0$,

$$\frac{P_n}{p_i} g_{x_i}' = -\frac{G_n G_n'}{x_i} - \frac{(1 - G_n)G_n'}{1 - x_i} + 2(1 - A_n)$$

记 $P(x) = -G_n G_n'(1 - x) - (1 - G_n)G_n' x + 2(1 - A_n)x(1 - x) = 0$, $p(0) < 0$, $2P(\frac{1}{2}) = -G_n' + 1 - A_n \geq 1 - A_n' - A_n = 0$. 得 $P(x) = 0$在$(0, \frac{1}{2})$恰有一个根, 于是 $a_1 = a_2 = \cdots = a_n$, 结论成立

若 $\underline{a}$是边界点

(i) $\underset{\sim}{a}$的分量全不为0,设有 $l(\geq 1)$ 个为$\frac{1}{2}$,不妨设 $a_{k+1}=\cdots=a_n=\frac{1}{2}$, $1\leq n-k=l\leq n-1$. 定义 h: $[0,\frac{1}{2}]^k\to R$

$$h(x_1,\cdots,x_k)=g(x_1,\cdots,x_k,\frac{1}{2},\cdots,\frac{1}{2})=\frac{1}{2}[1-\frac{1}{2}(2G_k)^{\frac{P_k}{P_n}}](2G_k')^{\frac{P_k}{P_n}}-[\frac{1}{2}+\frac{P_k(\frac{1}{2}-A_k)}{P_n}]^2$$

有

$$h(x_1,\cdots,x_k)\geq h(a_1,\cdots,a_k),\quad \forall(x_1,\cdots,x_k)\in[0,\frac{1}{2}]^k, \tag{8}$$

$0<a_i<\frac{1}{2}$,有$\nabla h(a_1,\cdots,a_k)=0$,计算可得

$$\frac{P_n}{p_i}h'_{x_i}=-\frac{1}{4}(4G_kG_k')^{\frac{P_k}{P_n}}\bullet\frac{1}{x_i}-\frac{1}{2}(2G_k')^{\frac{P_k}{P_n}}\frac{1}{1-x_i}+\frac{1}{4}(G_kG_k')^{\frac{P_k}{P_n}}\bullet\frac{1}{1-x_i}+(\frac{-2P_kA_k}{P_n}+1+\frac{P_k}{P_n})$$

记$Q(x)=\frac{1}{4}(4G_kG_k')^{\frac{P_k}{P_n}}(2x-1)-\frac{1}{2}(2G_k')^{\frac{P_k}{P_n}}\bullet x+(\frac{-2P_kA_k}{P_n}+1+\frac{P_k}{P_n})x(1-x)=0$. 有$Q(0)<0$, $4Q(\frac{1}{2})=-(2G_k')^{\alpha}-2A_k\alpha+1+\alpha$,其中$\alpha=\frac{P_k}{P_n}\in(0,1)$.

令$\tilde{Q}(\alpha)=-(2G_k')^{\alpha}-2A_k\alpha+1+\alpha$

$\tilde{Q}''(\alpha)=-(2G_k')^{\alpha}\ln^2(2G_k')<0$, $\tilde{Q}(\alpha)$在$[0,1]$严格上凸,又$\tilde{Q}(0)=0$, $\tilde{Q}(1)=2(1-A_k-G_k')\geq 2(1-A_k-A_k')=0$,得$\tilde{Q}(\alpha)>0$,即$Q(\frac{1}{2})=0$,从而$Q$ 恰有一根在$(0,\frac{1}{2})$内,便有

$$a_1=a_2=\cdots=a_k.$$

记$\tilde{h}(x)=h(x,\cdots,x)=\frac{1}{2}[1-\frac{1}{2}(2x)^{\alpha}][2(1-x)]^{\alpha}-[\frac{1}{2}+\alpha(\frac{1}{2}-x)]^2$

$\frac{1}{\alpha}\tilde{h}'(x)=(2x-1)[4x(1-x)]^{\alpha-1}-[2(1-x)]^{\alpha-1}+1+\alpha-2\alpha x$

记右边为$P(\alpha)$,则

$$P'(\alpha)=(2x-1)[4x(1-x)]^{\alpha-1}\ln[4x(1-x)]-[2(1-x)]^{\alpha-1}\ln[2(1-x)]+1-2x$$

$$P''(\alpha)=(2x-1)[4x(1-x)]^{\alpha-1}\ln^2[4x(1-x)]-[2(1-x)]^{\alpha-1}\ln^2(2-2x)<0.$$

故当$\alpha\in(0,1)$,有$P'(\alpha)>P'(1)=(2x-1)\ln[4x(1-x)]-\ln(2-2x)+1-2x$,记上式右边为$q(x)$,则

$$q'(x)=2\ln[4x(1-x)]+(2x-1)\frac{4-8x}{4x(1-x)}+\frac{2}{2(1-x)}-2$$

$$q''(x)=\frac{4x^3-5X^2+1}{[x(1-x)]^2}>0,\quad x\in(0,\frac{1}{2})$$

$q'(x)$严格递增, $q'(\frac{1}{2})=0$,有$q'(x)<0$, $q(x)$严格递减, $q(\frac{1}{2})=0$,有$q(x)>0$,即$P'(1)>0$,得$P'(\alpha)>0$, $P(\alpha)$严格递增, $P(\alpha)<P(1)=0$,即$\tilde{h}'(x)<0$,得$\tilde{h}(x)$严格递减, $h(a_1,\cdots,a_k)=\tilde{h}(a_1)>\tilde{h}(\frac{1}{2})=h(\frac{1}{2},\cdots,\frac{1}{2})$,这与(8)式矛盾,从而$a_1=\cdots=a_k=\frac{1}{2}$,结论成立.

(ii) $\underset{\sim}{a}$的 $l(\geq 1)$ 个分量为0,不妨设 $a_{k+1}=\cdots=a_n=0$, $1\leq n-k=l\leq n-1$. 定义φ: $[0,\frac{1}{2}]^k\to k$

$$\varphi(x_1, \cdots, x_k) = g(x_1, \cdots, x_k, 0, \cdots, 0) = (G_k')^{\frac{P_k}{P_n}} - \left(1 - \frac{P_k A_k}{P_n}\right)^2$$

$$\frac{P_n}{2p_i}\varphi'_{x_i} = \frac{-(G_k')^{\alpha}}{2(1-x_i)} + 1 - \alpha A_k \geq -(G_k')^{\alpha} + 1 - \alpha A_k$$

记 $\psi(\alpha) = -(G_k')^{\alpha} + 1 - \alpha A_k$. 在 $[0, 1]$ 严格上凸, 由 $\psi(0) = 0$, $\psi(1) = -G_k' + 1 - A_k = -G_k' + A_k' \geq 0$ 知 $\psi(\alpha) > 0$, $\alpha \in (0, 1)$, 即 $\varphi'_{x_i} > 0$, φ 严格递增, 有

$$\varphi(x_1, \cdots, x_k) \geq \varphi(0, \cdots, 0) = 0, \quad \forall (x_1, \cdots, x_k) \in [0, \frac{1}{2}]^k.$$

于是 $(a_1, \cdots, a_k) = (0, \cdots, 0)$. 结论成立 □

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Weighted Fan Ky Inequality and Its Refinements

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Abstract

In the present paper, a weighted Ky Fan's inequality is proved, and its two refinements are also given.

Keywords Fan Ky inequality.