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Some Properties of Meromorphic Functions with Maximal Quasi-Deficiency Sum*

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Abstract The property of meromorphic functions with maximal quasi-deficiency sum is discussed and some interesting results are obtained.

Keywords meromorphic function, deficient value, quasi-deficiency.

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1. Introduction and main results

In this paper, we use the notations as given in the Nevanlinna theory [1]. Let $S(r,f)$ denote arbitrary quantity that satisfies

$$S(r,f) = o(T(r,f)) \quad (r\to\infty),$$

which is not necersiarily the same at each time it occurs. Let f be a meromorphic function on the plane, and m be a positive integer. $\overline{n}_{m)}(r,a,f)$ and $\overline{n}_{(m}(r,a,f)$ denote the number of distinct zeros of $(f-a)$, whose multiplicities are less than and larger than m in the $|z|\le r$, respectively; the quantity $\overline{N}_{m)}(r,a,f)$ and $\overline{N}_{(m}(r,a,f)$ are defined in usual manner from $\overline{n}_{m)}(r,a,f)$ and $\overline{n}_{(m}(r,a,f)$. $n_{m)}(r,a,f)$ and $n_{(m}(r,a,f)$ denote the number of zeros, with due count of multiplicities, of $(f-a)$ whose multiplicities are less than and larger than m in the $|z|\le r$, respectively; the quantity $N_{m)}(r,a,f)$ and $N_{(m}(r,a,f)$ are defind in the usual manner from $n_{m)}(r,a,f)$ and $n_{(m}(r,a,f)$.

On the other hand, we define Nevanlinna's quasi- deficiency of f with respect to a complex number a (finite or infinite) by

$$\delta_{m)}(a,f) = 1 - \overline{\lim_{r\to\infty}}[\overline{N}_{m)}(r,a,f)/T(r,f)]. \tag{1}$$

It is known that from [2]

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$$0 \le \delta(a,f) \le \delta_{m)}(a,f) \le 1, \tag{2}$$

and

$$\sum \delta_{m)}(a,f) \le 2(m+1)/m. \tag{3}$$

In the present paper, we consider the case that f is of finite order and the equality holds in (3).

Theorem 1 *Let f be a transcendental meromorphic function with finite order, if the equality holds in* (3) *for some m, then for any $k \in N^+$ we have that*

(*i*) $\delta(a,f) = 0$ *for* $a \in C$.

(*ii*) $(m+1)\Theta(a,f) = m\,\delta_{m)}(a,f)$ *for* $a \in C$, *and* $\sum \Theta(a,f) = 2$.

(iii) $$\delta_{m)}(\infty, f^{(k)}) = \begin{cases} 1 & \text{as } m \le k, \\ \{k+u-[km/(m+1)]u\}/\{k+1-[km/(m+1)]u\} & \text{as } m > k, \end{cases}$$

where $u = \delta_{m)}(\infty, f)$.

Corollary 1 *Under the same assumptions as in Theorem* 1, *then*

$$\delta_{m)}(\infty, f^{(k)}) > k/(k+1). \tag{4}$$

Theorem 2 *Under the same assumption as in Theorem* 1, *then*

$$\delta(0, f/f') = 0. \tag{5}$$

2 Some Lemmas

Lemma 1 *let f be a transcendental meromorphic function with finite order, if the equality holds in* (3) *for some m, then we have that for any $a \in C$*

$$N_{m)}(r,a,f) = \overline{N}_{m)}(r,a,f) + S(r,f),$$

$$N_{(m}(r,a,f) = (m+1)\overline{N}_{(m}(r,a,f) + S(r,f), \tag{*}$$

$$N(r,a,f) = T(r,f) + S(r,f).$$

Proof First, it is easy to see that for any $a \in C$

$$\overline{N}(r,a,f) \le [m/(m+1)]\overline{N}_{m)}(r,a,f) + [1/(m+1)]N(r,a,f). \tag{6}$$

By the second fundamental theorem we have that

$$(p-2)T(r,f) < \sum_{i=1}^{p}\overline{N}(r,a_i,f) + S(r,f) = \sum_{i=1}^{p}[\overline{N}_{m)}(r,a_i,f) + \overline{N}_{(m}(r,a_i,f)] + S(r,f)$$

$$\le \sum_{i=1}^{p}\overline{N}_{m)}(r,a_i,f) + [1/(m+1)]\sum_{i=1}^{p}N_{(m}(r,a_i,f) + S(r,f),$$

i.e.,

$$[1/(m+1)]\sum_{i=1}^{p}[T(r,f) - N_{(m}(r,a_i,f)]$$

$$\leq \sum_{i=1}^{p} \overline{N}_{m)}(r, a_i, f) + [2 - mp/(m+1)]T(r,f) + S(r,f). \tag{7}$$

Obviously,

$$T(r,f) \geq N(r,a,f) + O(1) = N_{m)}(r,a,f) + N_{(m}(r,a,f) + O(1).$$

Hence, from (7) we get

$$\sum_{i=1}^{p} [N_{m)}(r, a_i, f) - \overline{N}_{m)}(r, a_i, f)]$$

$$\leq 2T(r,f) - m/(m+1) \sum_{i=1}^{p} [T(r,f) - \overline{N}_{m)}(r, a_i, f)] + S(r,f),$$

Thus

$$\varlimsup_{r\to\infty} \sum_{i=1}^{p} [N_{m)}(r, a_i, f) - \overline{N}_{m)}(r, a_i, f)]/T(r,f)$$

$$\leq 2 - m/(m+1) \sum_{i=1}^{p} (1 - \varliminf_{r\to\infty} [N_{m)}(r, a_i, f)/T(r,f)])$$

$$= 2 - m/(m+1) \sum_{i=1}^{p} \delta_{m)}(a_i, f).$$

By the assumption, the equality holds in (3), and let $p \to \infty$ we obtain that

$$\varlimsup_{r\to\infty} \sum ([N_{m)}(r,a,f) - \overline{N}_{m)}(r,a,f)]/T(r,f)) \leq 2 - [m/(m+1)][2(m+1)/m] = 0,$$

so that for any $a \in c$, $\varlimsup_{r\to\infty} [N_{m)}(r,a,f) - \overline{N}_{m)}(r,a,f)]/T(r,f) = 0$, i.e.,

$$N_{m)}(r,a,f) = \overline{N}_{m)}(r,a,f) + S(r,f).$$

Secondly, from (6) and the second fundamental theorem we can get

$$(p-2)T(r,f) \leq [m/(m+1)] \sum_{i=1}^{p} N_{m)}(r, a_i, f) + 1/(m+1) \sum_{i=1}^{p} N(r, a_i, f) + S(r,f).$$

It follows from this that

$$(p-2) \leq pm/(m+1) - [m/(m+1)] \sum_{i=1}^{p} \delta_{m)}(a_i, f)$$

$$+ p/(m+1) - [1/(m+1)] \sum_{i=1}^{p} \delta(a_i, f),$$

i.e.,

$$[m/(m+1)] \sum_{i=1}^{p} \delta_{m)}(a_i, f) + [1/(m+1)] \sum_{i=1}^{p} \delta(a_i, f) \leq 2,$$

we deduce from (3) that $[1/(m+1)]\sum\delta(a,f)\le 0$. Thus for any $a\in C$, $\delta(a,f)=0$, i.e., $N(r,a,f)=T(r,f)+S(r,f)$. Finaly, from (7) we have that

$$(m+1)(p-2)T(r,f)\le(m+1)\sum_{i=1}^{p}[\overline{N}_{m)}(r,a_i,f)+\overline{N}_{(m}(r,a_i,f)]+S(r,f).$$

It follows from this that

$$\sum_{i=1}^{p}[N_{(m}(r,a_i,f)-(m+1)\overline{N}_{(m}(r,a_i,f)]$$

$$\le m\sum_{i=1}^{p}\overline{N}_{m)}(r,a_i,f)+[2(m+1)-mp]T(r,f)+S(r,f).$$

Hence

$$\lim_{r\to\infty}\sum_{i=1}^{p}([N_{(m}(r,a_i,f)-(m+1)\overline{N}_{(m}(r,a_i,f)]/T(r,f)$$

$$\le m\sum_{i=1}^{p}\lim_{r\to\infty}[\overline{N}_{m)}(r,a_i,f)/T(r,f)]+2(m+1)-mp$$

$$=mp-m\sum_{i=1}^{p}\delta_{m)}(r,a_i,f)+2(m+1)-mp.$$

By the assumption, the equality holds in (3), and let $p\to\infty$ we get

$$\lim_{r\to\infty}\sum([N_{(m}(r,a,f)-(m+1)\overline{N}_{(m}(r,a,f)]/T(r,f))$$

$$\le 2(m+1)-m[2(m+1)/m]=0,$$

It is shows that for any $a\in C$

$$N_{(m}(r,a,f)=(m+1)\overline{N}_{(m}(r,a,f)+S(r,f).$$

This completes the proof of lemma 1. □

Lemma 2 *Let f be a meromorphic function with finite order, if the equality holds in* (3), *then*

$$\lim_{r\to\infty}[N(r,f)/T(r,f)]=1 \tag{8}$$

and

$$\lim_{r\to\infty}[\overline{N}(r,f)/T(r,f)]=1-[m/(m+1)]u. \tag{9}$$

Proof (8) is an immediate consequence of (*). Next, it is easy to see that from (6)

$$\overline{\lim_{r\to\infty}}[\overline{N}(r,f)/T(r,f)]\le 1-[m/(m+1)]u, \tag{10}$$

but we can get that from (*)

$$\varliminf_{r\to\infty}[\overline{N}(r,f)/T(r,f)] = \varliminf_{r\to\infty}([\overline{N}_{m)}(r,f) + \overline{N}_{(m}(r,f)]/T(r,f))$$

$$= \varliminf_{r\to\infty}([\overline{N}_{m)}(r,f) + 1/(m+1)N_{(m}(r,f) + S(r,f)]/T(r,f))$$

$$= \varliminf_{r\to\infty}([\overline{N}_{m)}(r,f) + 1/(m+1)N(r,f) - 1/(m+1)N_{m)}(r,f) + S(r,f)]/T(r,f))$$

$$= \varliminf_{r\to\infty}(m/(m+1)\overline{N}_{m)}(r,f) + 1/(m+1)T(r,f) + S(r,f)]/T(r,f))$$

$$\geq 1 - [m/(m+1)]u. \tag{11}$$

Combining (10) with (11) we get (9).

Lemma 3[3] *Let f be a meromorphic function, if the equality holds in* (3), *then*

$$T(r,f^{(k)}) \sim (k + 1 - [km/(m+1)]u)T(r,f). \tag{12}$$

3. The proof of theorem

Proof of Theorem 1 (i) is an immediate consequance of (*).

(ii) By (6) we have that for any $a \in C$,

$$\overline{N}(r,a,f) \leq [m/(m+1)]\overline{N}_{m)}(r,a,f) + [1/(m+1)]N(r,a,f).$$

Hence

$$\varlimsup_{r\to\infty}[\overline{N}(r,a,f)/T(r,f)] \leq [m/(m+1)]\varlimsup_{r\to\infty}[\overline{N}_{m)}(r,a,f)/T(r,f)] + 1/(m+1),$$

i.e.,

$$m\delta_{m)}(a,f) \leq (m+1)\Theta(a,f). \tag{13}$$

On the other hand, we can deduce from (*) that

$$T(r,f) + S(r,f) = N(r,a,f) = N_{m)}(r,a,f) + N_{(m}(r,a,f)$$

$$= (m+1)\overline{N}(r,a,f) - m\overline{N}_{m)}(r,a,f) + S(r,f),$$

i.e.,

$$1 + m[\overline{N}_{m)}(r,a,f)/T(r,f)] = (m+1)[\overline{N}(r,a,f)/T(r,f)] + S(r,f)/T(r,f).$$

Hence

$$1 + m\varlimsup_{r\to\infty}[\overline{N}_{m)}(r,a,f)/T(r,f)] \leq (m+1)\varlimsup_{r\to\infty}[\overline{N}(r,a,f)/T(r,f)],$$

i.e.,

$$(m+1)\Theta(a,f) \leq m\delta_{m)}(a,f). \tag{14}$$

Combining (13) with (14) deduce that for any $a \in C$

$$(m+1)\Theta(a,f) = m\delta_{m)}(a,f).$$

It follows from this and (3) that

$$\sum \Theta(a,f) = [m/(m+1)] \sum \delta_{m)}(a,f) = 2.$$

(iii) It is easy to see from $N_{m)}(r, f^{(k)}) = 0$ $(m \le k)$ that

$$\delta_{m)}(\infty, f^{(k)}) = 1, \qquad m \le k.$$

When $m > k$, since

$$\overline{\lim_{r\to\infty}}[\overline{N}_{m)}(r,f^{(k)})/T(r,f^{(k)})] \le \overline{\lim_{r\to\infty}}[\overline{N}_{m)}(r,f)/T(r,f^{(k)})]$$

$$\le \overline{\lim_{r\to\infty}}[\overline{N}_{m)}(r,f)/T(r,f)]\ \overline{\lim_{r\to\infty}}[T(r,f)/T(r,f^{(k)})]$$

$$= (1-u)/(k+1-[km/(m+1)]u). \tag{15}$$

Next, note that $m > k$ and $N_{m)}(r,a,f) = \overline{N}_{m)}(r,a,f) + S(r,f)$, it follows from this that

$$\overline{N}_{m)}(r,a,f^{(k)}) = \overline{N}_{m)}(r,a,f) + S(r,f).$$

Hence

$$\underline{\lim_{r\to\infty}}[\overline{N}_{m)}(r,f^{(k)})/T(r,f^{(k)})] = \underline{\lim_{r\to\infty}}[\overline{N}_{m)}(r,f) + S(r,f)]/T(r,f^{(k)})$$

$$\ge \underline{\lim_{r\to\infty}}[\overline{N}_{m)}(r,f)/T(r,f)]\ \underline{\lim_{r\to\infty}}[T(r,f)/T(r,f^{(k)})].$$

By Lemma 3 we get

$$\underline{\lim_{r\to\infty}}[\overline{N}_{m)}(r,f^{(k)})/T(r,f^{(k)})] \ge (1-u)/\{k+1-[km/(m+1)]u\}. \tag{16}$$

Combining (15) with (16) we have that

$$\underline{\lim_{r\to\infty}}[\overline{N}_{m)}(r,f^{(k)})/T(r,f^{(k)})] = (1-u)/\{k+1-[km/(m+1)]u\}.$$

Hence

$$\delta_{m)}(\infty, f^{(k)}) = \{k+u-[km/(m+1)]u\}/\{k+1-[km/(m+1)]u\}. \tag{17}$$

Proof of Corollary 1 Let

$$g(u) = \{k+u-[km/(m+1)]u\}/\{k+1-[km/(m+1)]u\}, \qquad 0 \le u \le 1. \tag{18}$$

Then

$$g'(u) = \frac{\{[1-km/(m+1)]/(k+1-[km/(m+1)]u) + [km/(m+1)](k+u-[km/(m+1)u])\}}{\{(k+1-[km/(m+1)]u)\}}$$

$$= [1+k/(m+1)]/(k+1-[km/(m+1)]u), \qquad 0 < u < 1.$$

Thus $g(u)$ is a increasing function on the $[0,1]$, hence

$$g(u) > g(0) = k/(k+1).$$

From (17) we deduce immediately that (4) is true.

Proof of Theorem 2 Let $g = 1/f$, obviously

$$\sum \delta_{m)}(a,f) = \sum \delta_{m)}(a,g), \text{ and } \delta_{m)}(0,f) = \delta_{m)}(\infty,g).$$

Therefore

$$\sum \delta_{m)}(a,g) = \sum \delta_{m)}(a,f) = 2(m+1)/m,$$

by Lemma 3 we get

$$T(r,g')\sim \{2-[m/(m+1)]\delta_{m)}(\infty,g)\}T(r,g) = \{2-[m/(m+1)]\delta_{m)}(0,f)\}T(r,f). \tag{19}$$

Since $g' = -f'/f^2$, hence $T(r,g') \le T(r,f'/f) + T(r,f) + S(r,f)$.

From (19) we deduce that for sufficiently large r

$$T(r,f'/f) \ge T(r,g') - T(r,f) - S(r,f) \ge \{1-[m/(m+1)]\delta_{m)}(0,f)\}T(r,f). \tag{20}$$

Noting that f is a meromorphic function with finite order, hence

$$m(r,f'/f) = S(r,f). \tag{21}$$

Combining (20) with (21) we get that

$$\begin{aligned}\delta(0,f/f') &= \underline{\lim}_{r\to\infty}[m(r,f'/f)/T(r,f'/f)]\\ &\le \underline{\lim}_{r\to\infty} S(r,f)/([1-(m/(m+1))\delta_{m)}(0,f)]T(r,f))\\ &\le \underline{\lim}_{r\to\infty} S(r,f)/([1/(m+1)]T(r,f)) = 0.\end{aligned}$$

This completes the proof of theorem 2.

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具最大拟亏量和的亚纯函数的某些性质

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摘 要

本文讨论具最大拟亏量和的亚纯函数的性质, 得到一些有趣的结果