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On Spectral Charactercations of Isoparametric Hypersurfaces in S^4 *

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Abstract In this paper, we prove that, as constant mean curvature hypersurfaces, the isoparametric hypersurfaces M in $S^4(1)$ can be characterized by their (strongly) spectrum.

Keywords isoparametric hypersurfaces, spectram.
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1. Introduction

The isospectral problem is an important problem in the theory of Riemannian geometry. Generally speaking, the answer to the isospectral problems is negative. The first counter example was constructed by J. Milnor[8]. Therefore, the study of this problem is divided into two directions. One is to construct new counter examples, which have been studied by Vigneras[10] and Ikeda[7]. The other is to give an affirmative answer for some special Riemannian manifolds. In this respect, Berger[1], Patodi[9] have done some fundamental and profound works.

In this paper, we study the latter problem for the hypersurfaces in sphere S^4 with constant mean curvature. Let M be a smooth, compact and oriented Riemannian manifold of dimension n. By $\wedge^p(M)$ we denote the space of differential forms of degree p with real coefficients, $p = 0, 1, \cdots, n$. $\mathrm{Spec}^p(M)$ the spectrum of the Laplace operator acting on $\wedge^p(M)$. Donnelly[4] proved that the totally geodesic submanifold in n dimensional sphere can be characterized by the spectrum $\mathrm{Spec}^0(M)$ and its minimality. Hasegawa[6] showed that there are many concrete minimal submanifolds in sphere, such as Veronese manifold, can be characterized by their spectrum. Lu Zhiqin and Chen Zhihua[5] proved that if M is a minimal hypersurface in $S^{n+1}(1)$, $\mathrm{Spec}^p(M) = \mathrm{Spec}^p(S^n(1))$ and $(n, p) \neq (6, 1)$ or $(6, 5)$, then $M = S^n(1)$. We know that there are many isoparametric hypersurfaces in $S^4(1)$, which have been characterized by constant mean and scalar curvatures in $S^4(1)$ (see [2]). It's natural to ask: whether these hypersurfaces can be characterized by their spectrum? The answer is:

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Theorem *Let M be a hypersurface in $S^4(1)$ with constant mean curvature h ($h \neq 0$), M_0 be an isoparametric hypersurface in $S^4(1)$ with the same mean curvature h. If* $\operatorname{Spec}^p(M) = \operatorname{Spec}^p(M_0)$ $(p = 0, 1)$, *then* $M = M_0$.

2 Preliminaries

Suppose M is a 3-dimensional compact, connected, smooth hypersurface in $S^4(1)$. Let $R, \overline{R}, \rho$ denote the Riemann curvature tensor, Ricci curvature tensor and scalar curvature of M respectively. R_{ijkl} denote components of R (a similarly way of $\overline{R}$). Gauss equations reads

$$R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + h_{ik}h_{jl} - h_{il}h_{jk}, \tag{1}$$

where δ_{ij} is Kronecher symbol, h_{ij} is component of the second fundamental forms of M in $S^4(1)$. For a fixed point $x_0 \in M$, we can choose a proper orthonormal frame e_1, e_2, e_3 such that (h_{ij}) are diagonal at x_0, say

$$h_{ij} = \lambda_i\delta_{ij}.$$

Let $h = \sum h_{ii} = \lambda_1 + \lambda_2 + \lambda_3$ be mean curvature of M, $S = \sum h_{ij}^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$ the square length of the second fundamental form, then we have

$$R_{ijkl} = (1 + \lambda_i\lambda_j)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \tag{2}$$

$$\widetilde{R}_{ij} = [2 + h\lambda_i - \lambda_i\lambda_j]\delta_{ij}, \tag{3}$$

$$\rho = 6 + h^2 - S. \tag{4}$$

Because M is compact, for $p = 0, 1, 2, 3$, we let

$$\operatorname{Spec}(M) = \{0 \leq \lambda_{0,p} \leq \lambda_{1,p} \leq \cdots \uparrow + \infty\}.$$

For these discrete eigenvalues, we have the Minakshisundaram-Pleijel asymptotic formula

$$\sum_{i=0}^{\infty} e^{-\lambda_{i,p}t} \sim (4\pi t)^{-\frac{3}{2}}(a_{0,p} + a_{1,p}t + a_{2,p}t^2 + \cdots), \qquad (t \to 0^+)$$

here the coefficients $a_{k,p}$, $k = 0, 1, 2$ were calculated by Patodi in [9] as follows:

$$a_{0,p} = \binom{3}{p}\operatorname{vol}(M), \tag{5}$$

$$a_{1,p} = \left(\frac{1}{6}\binom{3}{p} - \binom{1}{p-1}\right)\int_M \rho\, dv, \tag{6}$$

$$a_{2,p} = \int_M (c_1(p)\rho^2 + c_2(p)|\widetilde{R}|^2 + c_3(p)|R|^2)\, dv, \tag{7}$$

where dv denotes the volume element of M and

$$c_1(p) = \frac{1}{72}\binom{3}{p} - \frac{1}{6}\binom{1}{p-1};$$

$$c_2(p) = -\frac{1}{180}\binom{3}{p} + \frac{1}{2}\binom{1}{p-1};$$

$$c_3(p) = \frac{1}{180}\binom{3}{p} - \frac{1}{12}\binom{1}{p-1},$$

here $\binom{l}{q}$ is understood to be zero when $l < 0$ or $q < 0$ or $l < q$.

Finally, we still need a result below:

Theorem A (Chang[2] or Cheng and Wan[3]) *The hypersurface in* $S^4(1)$ *with constant mean and scalar curvature are isoparametric.*

3. The Proof of Theorem

Because M is a hypersurface in $S^4(1)$ with constant mean curvature h from (2) - (4) we get

$$|R|^2 = 2S^2 - 2\sum \lambda_i^4 + 4h^2 - 4S + 12, \tag{8}$$

$$|\widetilde{R}|^2 = h^2 S + \sum \lambda_i^4 - 2h\sum \lambda_i^3 + 12 - 4S. \tag{9}$$

where $\sum \lambda_i^4$, $\sum \lambda_i^3$ are smooth coeffecients globally defined functions on M. Since M_0 is an isoparametric in $S^4(1)$ with constant mean curvature h. We know that M_0 has the constant principal curvatures λ_i^0 $(1 \leq i \leq 3)$. Let $\rho_0, \widetilde{R}_0, R_0$ and S_0 denote respectively the scalar curvature, Ricci curvature tensor, Curvature tensor and the square of the length of the second fundamental form of M_0. Then $\rho_0 = 6 + h^2 - S_0$, $|\widetilde{R}_0|^2 = 2h^2 S_0 + \sum (\lambda_i^0)^4 - 2h\sum (\lambda_i^0)^3 + 12 - 4S_0$, $|R|^2 = 2S_0^2 - 2\sum (\lambda_i^0)^4 + 4h^2 - 4S_0 + 12$ and $S_0 = \sum (\lambda_i^0)^2$. Let $a_{k,p}$ and $a_{k,p}^0$ be the coefficients of the asymptotic expansion of Minakshisundaram - Pleijel corresponding to M and M_0 respectively. Since $\mathrm{Spec}^p(M) = \mathrm{Spec}^p(M_0)$ for $p = 0, 1$, we have $a_{k,p} = a_{k,p}^0$ for $k = 0, 1, 2$ from the asymptotic expansion formula. Thus, by (5) - (7), we have

$$\mathrm{vol}(M) = \mathrm{vol}(M_0), \tag{10}$$

$$\int_M \rho dv = \int_{M_0} \rho_0 dv_0, \tag{11}$$

$$\int_M (c_1(p)\rho^2 + c_2(p)|\widetilde{R}|^2 + c_3(p)|R|^2)dv$$

$$= \int_{M_0} (c_1(p)\rho_0^2 + c_2(p)|\widetilde{R}_0|^2 + c_3(p)|R_0|^2)dv_0. \tag{12}$$

Here we have used $\frac{1}{6}\binom{3}{p} \neq \binom{1}{p-1}$ for any $p = 0, 1, 2, 3$ in (3.5). Substituting (4), (8) and (9) into (12) and making use of (10), (11), we have for $p = 0, 1$

$$\int_M [(c_1(p) + 2c_3(p))S^2 + (c_2(p) - 2c_3(p))\sum \lambda_i^4 - 2c_2(p)h\sum \lambda_i^3]dv_0$$

$$= \int_{M_0} [(c_1(p) + 2c_3(p))S_0^2 + (c_2(p) - 2c_3(p))\sum (\lambda_i^0)^4 - 2c_2(p)h\sum (\lambda_i^0)^3]dv_0. \tag{13}$$

and

$$\int_M S\,dM = \int_{M_0} S_0\,dM_0. \tag{14}$$

Since M and M_0 are 3- dimensional hypersurfaces in $S^4(1)$ with constant mean curvature h, direct computation shows that:

$$\sum \lambda_i^4 = \frac{1}{6}h^4 + \frac{4}{3}h\sum \lambda_i^3 - h^2 S + \frac{1}{2}S^2,$$

and

$$\sum (\lambda_i^0)^4 = \frac{1}{6}h^4 + \frac{4}{3}h\sum (\lambda_i^0)^3 - h^2 S + \frac{1}{2}S_0^2.$$

Put them in (13), we obtain

$$(c_1(p) + \frac{1}{2}c_2(p) + c_3(p))\left(\int_M S^2 dM - \int_{M_0} S_0^2 dM_0\right)$$
$$- \left(\frac{2}{3}c_2(p) + \frac{8}{3}c_3(p)\right)h\left(\int_M \sum \lambda_i^3 dM - \int_{M_0}\sum (\lambda_i^0)^3 dM_0\right) = 0, \tag{15}$$
$$p = 0, 1.$$

For $h \neq 0$ and

$$\det\begin{pmatrix} c_1(0) + \frac{1}{2}c_2(0) + c_3(0) & -\frac{2}{3}c_2(0) - \frac{8}{3}c_3(0) \\ c_1(0) + \frac{1}{2}c_2(1) + c_3(1) & -\frac{2}{3}c_2(1) - \frac{8}{3}c_3(1)\end{pmatrix} = -\frac{1}{9}\left(\frac{1}{72} + \frac{1}{360}\right) \neq 0,$$

the equations (15) exists unique solution:

$$\int_M S^2 dM = S_0^2 \mathrm{vol}(M_0),$$
$$\int_M \sum \lambda_i^3 dM = \sum (\lambda_i^0)^3 \mathrm{vol}(M_0). \tag{16}$$

By using (14), the first equation in (16) and Schwarz inequality, we get

$$S_0 \mathrm{vol}(M_0) = \int_M S\,dM \leq \left(\int_M S^2 dM\right)^{\frac{1}{2}}\left(\int_M dM\right)^{\frac{1}{2}} = S_0 \mathrm{vol}(M_0).$$

Hence

$$S = S_0.$$

Thus M is a constant mean curvature hypersurface in $S^4(1)$ with constant scalar curvature. Theorem A implies that M is an isoparametric hypersurface. Therefore, M has constant principal curvatures and satiefies

$$\sum_i \lambda_i^k = \sum_i (\lambda_i^0)^k, \quad k = 1, 2, 3.$$

Regardless of a difference of permutational order, it is easy to see that $\lambda_i = \lambda_i^0$, $i = 1, 2, 3$. This proves $M = M_0$.

References

[1] M. Berger, P. Gauduchon et E. Mazet, *Le Spactre d'une Variete riemannienne*, Lect. Notes in Math. Springer-Verlag, **194** (1974).

[2] S. Chang, *A closed hypersurface with constant scalar and mean curvature in S is isoparametric*, Comm in Analysis and Geometry, **1** (1993), 71-100.

[3] Q. M. Cheng and X. R. Wan, *Hypersurfaces of space form M (c) with constant mean curvature*, Lect. Notes 2, First MSJ, International Research Institute on Geometry and Global Analysis, Japan 1993, 327-347.

[4] H. Donnelly, *Spectral invariants of the second variation operator*, Illinois J. Math., **21** (1977), 185-189.

[5] Z. Q. Lu and Z. H. Chen, *On the spectrum of the laplacian on minimal hypersurface of a sphere*, Chinese Advances in Math., **21**: **3** (1992).

[6] T. Hasegawa, *Spectral geometry of closed minimal submanifolds in a space form, real and complex*, Kodai Math. J., **3** (1980), 224-252.

[7] A. Ikeda, *On spherical space forms which are isospectral but not isometric*, J. Math. Soc., Japan **35** (1983), 437-404.

[8] J. Milnor, *Eigenvalues of the Laplace operator of certain manifolds*, P. N. A. S., **51**: **4** (1964), 542.

[9] U. K. Padati, *Curvature and the fundamental solution of the heat operator*, *J. Indian Math. Soc.*, **34** (1970), 269-285.

[10] *M. Vigneras, Variete riemannianne isospectrales et nonisotriques*, Ann. of Math., **112** (1980), 21-32.

S^4中等参超曲面的谱刻画

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摘　　要

本文证明了如果S^4中的具常平均曲率h的超曲面M与其具平均曲率h的等参超曲面M_0(强)等谱, 则$M = M_0$.