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On Congruence Properties of Stirling-type Pairs*

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Abstract We prove some congruence relations satisfied by integral Stirling-type pairs. The results settle a question posed by Hsu [5]. In particular, they extend known congruence properties of Stirling numbers of the first kind and the second kind.

Keywords Stirling numbers, congruence.

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1. Introduction

Let $f(t)$ and $g(t)$ be two formal power series such that $f(0) = g(0) = 0$ and $f(g(t)) = g(f(t)) = t$, i.e., f and g are reciprocal. A Stirling-type pair, is usually defined to be the coefficients of the following power type expansions

$$\frac{(f(t))^k}{k!} = \sum_{n\geq k} A_1(n,k)\frac{t^n}{n!}, \qquad \frac{(g(t))^k}{k!} = \sum_{n\geq k} A_2(n,k)\frac{t^n}{n!}.$$

It is known that a Stirling-type pair $A_1(n,k)$, $A_2(n,k)$ may also be characterized equivalently by the orthogonality of $A_1(n,k)$ and $A_2(n,k)$; by the Lagrange inversion formula between $f(t)$ and $g(t)$; or by the Schlömilch-type formula representing $A_1(\cdot,\cdot)$ linearly in terms of $A_2(\cdot,\cdot)$, and vise versa. See [5] for details and other properties of Stirling-type pairs.

Let $f(t) = \sum_{n\geq 1} a_n t^n/n!$, $g(t) = \sum_{n\geq 1} b_n t^n/n!$ with $a_1, a_2, \cdots \in \mathbf{Z}$, the ring of integers. Then it is known by Lagrange's inversion formula that $b_1, b_2, \cdots \in \mathbf{Z}$, and vise versa. In this case, we have by Hurwitz's lemma [3] that

$$(f(t))^k \equiv (g(t))^k \equiv 0 \pmod{k!}.$$

Thus we have $A_1(n,k)$, $A_2(n,k) \in \mathbf{Z}$.

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As a unified generalization of Stirling numbers of the first and second kind, the concept of Stirling-type pair may be used to cover many inverse relations of bivariate sequences, e.g., the Stirling numbers of the first and second kinds, tangent and arctangent numbers [3], Stirling-Comtet numbers [6] etc. Integral Stirling-pairs enjoy some nice arithmetic properties. In [4], it was shown that for a prime p and k, such that $1 < k < p \leq 2k - 1$ there hold the system of congruences

$$A_1(p+j, k+j) = A_2(p+j, k+j) \equiv 0 \pmod{p},$$

where $0 \leq j \leq p - k$.

It was also guessed that (1) may not be true under the sole restriction $1 < k < p$ and $0 \leq j \leq p - k$.

Here we shall answer this question in the negative, and prove some general congruences for integral Stirling-type pairs. In particular, these results reduce to new congruence propertites for Stirling numbers of the first kind and the second kind.

2 Main results

Recall that the Bell polynomial $\Phi_n = \Phi_n(a_1, a_2, \cdots)$ may be defined by $\Phi_0 = 1$ and

$$\sum_{n \geq 0} \Phi_n \frac{t^n}{n!} = \exp\left(a_1 t + \frac{t^2}{2!} + \cdots\right). \tag{2}$$

Defferentiating both sides of (2) with respect to t, and compare the coefficients of t^n we get

$$\Phi_{n+1} = \sum_{j=0}^{n} \binom{n}{j} \Phi_{n-j} a_{j+1}. \tag{3}$$

Notice that for $p \not| r$, $\binom{pn}{r} \equiv 0 \pmod{p}$, and $\binom{pn}{pi} \equiv \binom{n}{i} \pmod{p}$ for $n, i \in \mathbf{Z}$. When replace n by pn in (3), we get

$$\Phi_{pn+1} \equiv \sum_{j=0}^{n} \binom{n}{i}. \tag{4}$$

The following lemma is due to Carlitz [2], which is a generalization of a previous result of Bell [1].

Lemma *Let p be a prime. Then the following congruences hold.*

$$\Phi_{p+n} \equiv (a_1^p + a_p)\Phi_n + \sum_{j=1}^{n} \binom{n}{j} a_{p+j} \Phi_{n-j} \pmod{p}, \tag{5}$$

$$\Phi_{p^r} \equiv a_1^{p^r} + a_p^{p^{r-1}} + \cdots + a_{p^r} \pmod{p}. \tag{6}$$

Theorem 1 *Let p be a prime and* $1 < k < p$.

$$A_1(p,k) \equiv A_2(p,k) \equiv 0 \pmod p,$$

$$A_1(p,1) \equiv a_p, \quad A_1(p,p) \equiv a_1;\ A_2(p,1) \equiv b_p,\ A_2(p,p) \equiv b_1.$$

Proof Let z be a complex indeterminate. Define the polynomial $T_n(z)$ by

$$\exp zf(t) = \sum_{n \ge 0} T_n(z) \frac{t^n}{n!}.$$

Then it may be deduced from (1) that

$$\exp zf(t) = \sum_{n \ge 0} T_n(z) \frac{t^n}{n!} = \sum_{n,k} A_1(n,k) z^k \frac{t^n}{n!},$$

$$T_n(z) = \sum_{k=0}^{n} A_1(n,k) z^k = \sum_{k=1}^{n} A_1(n,k) z^k,$$

where the term for $k = 0$ has been deleted since $A_1(n,0) = 0$. Moreover,

$$T_n(z) = \Phi_n(za_1, za_2, \cdots).$$

Thus we have from (5) by letting $n = 0$ that $A_1(p,k) \equiv 0 \pmod p$. The remaining parts of the theorem may be proved by considering the function $g(t)$ and $A_2(n,k)$, or just by symmetry. □

Theorem 2 *Let p be a prime. Then for k with* $1 < k < p$, *we have*

$$A_1(p+j, k+j) \equiv A_2(p+j, k+j) \equiv 0 \pmod p,$$

where $0 \le j \le p - k$.

Proof By (5), we have

$$T_{p+j}(z) \equiv (a_1^p z^p + a_p z) \sum_{i=1}^{j} A_1(j,i) z^i + \sum_{r=1}^{j} \binom{j}{r} a_{p+j} \sum_{s=1}^{j-r} A_1(j-r, s) z^s,$$

then for fixed j, we have, by comparing the coefficients of t^k with $1 \le k \le p + j$ that if $j + 1 < k < p$

$$A_1(p+j, k) \equiv 0 \pmod p.$$

The proof may be completed by symmetry and by an obvious transformation.

Finally we note that it may be deduced from (6) that

Theorem 3 *Let p be a prime. If* $r \ge 0$, *then*

$$A_1(p^r, k) \equiv A_2(p^r, k) \equiv 0 \pmod p,\ k \ne p^r, p^{r-1}, \cdots, p, 1.$$

Remark Theorem 2 answers a question of Hsu [5]. If $f(t) = \ln(1+t)$, $g(t) = e^t - 1$, then Theorem 2 reduces to known congruences for Stirling numbers of the first kind and the second kind, see [6]. In this case, Theorem 3 extends known congruences for Stirling numbers of the first and second kinds, see Howard [4] for other congruences.

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关于广义 Stirling 数偶的同余性

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摘　　要

本文证明了广义 Stirling 数偶的一些同余性质, 从而回答了文[5]中的一个猜测. 这些结果做为特例推广了已知的关于两类 Stirling 数的同余性质.