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A Remark on the Inverse of Principal Matrices by Implicit LU Factorization*

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Abstract We prove the inverting formular of the principal submatrix on a symmetric strongly nonsingular matrix by the implicit LU factorization algorithm.

Keywords inverting matrix, principal submatrix, symmetric strongly nonsingular matrix, implicit LU factorization algorithm.

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The following theorem is stated in [1] concerning the inverse of the principal submatrix of a given matrix A but with a incorrect proof.

Theorem[1] *Let $p_1, p_2, \cdots, p_k$ be the first k ($k \le n$) search vectors generated by the implicit LU factorization algorithm on the n by n symmetric strongly nonsingular matrix A. Then*

$$\sum_{i=1}^{k} \frac{p_i p_i^T}{a_i^T p_i} = \begin{pmatrix} (A^{k,k})^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \tag{1}$$

where $A^{k,k}$ is the kth principal submatrix of A.

In this note we prove this theorem.

Proof Denote $A \equiv (a_{ij}) = (a_1, a_2, \cdots, a_n)^T$, $A^i = (a_1, a_2, \cdots, a_i)$. We procced by induction. For $k = 1$, from the implicit LU factorization algorithm [1], $p_1 = e_1 \in R^n$. It follows that

$$\frac{p_1 p_1^T}{a_1^T p_1} = \begin{pmatrix} (a_{11})^{-1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (A^{11})^{-1} & 0 \\ 0 & 0 \end{pmatrix},$$

the theorem is true for $k = 1$.

For $k = 2$, $H_2 = I - a_1 p_1^T / a_1^T p_1 = (h_{21}, e_2, e_3, \cdots, e_n)$, where $h_{21} = (0, -a_{12}/a_{11}, -a_{13}/a_{11}, \cdots, -a_{1n}/a_{11})^T$, and $p_2 = H_2^T e_2 = (-a_{21}/a_{11}, 1, 0, \cdots, 0)^T$. Thus $\sum_{i=1}^{2} p_i p_i^T / a_i^T p_i$ is of the following forms $\begin{pmatrix} B_2 & 0 \\ 0 & 0 \end{pmatrix}$, where $B_2 \in R^{2\times 2}$. It is enough to prove that

$$\sum_{i=1}^{k} \frac{p_i p_i^T}{a_i^T p_i} \begin{pmatrix} A^{k,k} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} \tag{2}$$

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holds for $k = 2$. Since $\frac{p_1 p_1^T}{a_1^T p_1}\begin{pmatrix} A^{2,2} & 0 \\ 0 & 0 \end{pmatrix} = (e_1, (a_{12}/a_{11})e_1, 0, \cdots, 0)$ and $p_2^T a_1 = e_2^T H_2 a_1 = 0$, then

$$\frac{p_2 p_2^T}{a_2^T p_2}\begin{pmatrix} A^{2,2} & 0 \\ 0 & 0 \end{pmatrix} = p_2/a_2^T p_2 (p_2^T A^2, 0) = (0, (-a_{12}/a_{11})e_1 + e_2, 0, \cdots, 0). \tag{3}$$

Hence when $k = 2$, (2) is true.

Assume that the theorem is true for $i \le k$, we prove that it will still be valid for $i = k + 1$. Denote $A^{k+1,k+1} = \begin{pmatrix} A^{k,k} & \overline{a}_{k+1} \\ \overline{a}_{k+1}^T & a_{k+1,k+1} \end{pmatrix}$, and from assuming the validity for $i \le k$, then

$$\sum_{i=1}^{k+1} \frac{p_i p_i^T}{a_i^T p_i}\begin{pmatrix} A^{k+1,k+1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=1}^{k} \frac{p_i p_i^T}{a_i^T p_i}\begin{pmatrix} 0 & \overline{a}_{k+1} & 0 \\ \overline{a}_{k+1}^T & a_{k+1,k+1} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{p_{k+1} p_{k+1}^T}{a_{k+1}^T p_{k+1}}\begin{pmatrix} A^{k+1,k+1} & 0 \\ 0 & 0 \end{pmatrix}$$

$\equiv I + II + III$. Since the last $(n - i)$ $(i \le k)$ components of p_i are zero, so $II = \sum_{i=1}^{k} \frac{p_i p_i^T}{a_i^T p_i}\begin{pmatrix} 0 & \overline{a}_{k+1} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} (A^{k,k})^{-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & \overline{a}_{k+1} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & (A^{k,k})^{-1}\overline{a}_{k+1} & 0 \\ 0 & 0 & 0 \end{pmatrix}$, where $(A^{k,k})^{-1}\overline{a}_{k+1} (\in R^k)$ lies in the $(k + 1)$ th column. Recall that [1]

$$H_{k+1} = I - \begin{pmatrix} I_k & 0 \\ \overline{A}_k (A^{k,k})^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\overline{A}_k (A^{k,k})^{-1} & I_{n-k} \end{pmatrix},$$

where $\overline{A}_k$ is the matrix comprising the last $n - k$ rows of A^k. Since $p_{k+1} = H_{k+1}^T e_{k+1} = (-\overline{a}_{k+1}^T (A^{k,k})^{-1}, 1, 0, \cdots, 0)^T$, thus $II = \begin{pmatrix} 0 & -p_{k+1} + e_{k+1} & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Applying $p_{k+1}^T a_j = 0$, $j < k + 1$, then $III = (0, p_{k+1}, 0)$, where the vector p_{k+1} lies in the $(k + 1)$ th column. From above, we have

$$\sum_{i=1}^{k+1} \frac{p_i p_i^T}{a_i^T p_i}\begin{pmatrix} (A^{k+1,k+1}) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_k & -p_{k+1} + e_{k+1} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & p_{k+1} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} I_{k+1} & 0 \\ 0 & 0 \end{pmatrix}.$$

The proof is complete.

Remark From (3) it may be seen that the statement 'From (6.99) it follows that $p_i u_i^T$ is a matrix whose elements are all zero, save that on the intersection of the ith row with the ith column, which is equal to $a_i^T p_i$.' in [1] is wrong.

References

[1] J. Abaffy and E. Spedicato, *ABS Projection Algorithm: Mathematical Techniques for Linear and Nonlinear Equations*, John Wiley & Sons, New York, 1989.