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On Categoricities of (Co-)Reflexivity and Their Applications*

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Abstract In this paper, we study some categoricities of reflexivity and coreflexivity. The main results are as follows: (1). $\mathbf{A}$ and $\mathbf{C}$ are a pair of equivalent categories (2). Reflexivity and coreflexivity can be extended as mentioned in the paper. Those can be used to simplify and generalize some results of [1-4].

Keywords reflexivity, coreflexivity, algebras, coalgebras
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1. Preliminaries

The notions of (co-)reflexivity were proposed by E. J. Taft and D. E. Radford in different way in 1973 (see [1-3]). E. J. Taft mainly considered relations of dualization, while D. E. Radford worked out some conditions of finiteness. Now, those are still basic objects of study. We are mainly interested in categoricities of (co-)reflexivity and their applications in this paper.

Throughout the paper everything take place over fixed field $\mathbf{k}$. We assume readers know basic theory of Hopf algebras, so we directly use the notation and terminology of [4].

Let $\mathbf{C}$ be the category of coalgebras, $\mathbf{A}$ the category of algebras. They are non-full subcategories of $\mathbf{k}$-**mod**. Recall that a morphism f in $\mathbf{k}$-**mod** is isomorphic, monic, epic iff f of the underlying set is bijective, injective, surjective respectively. However, there is little difference between epics and surjections in $\mathbf{A}$, so we work with $\widetilde{\mathbf{A}}, \widetilde{\mathbf{C}}$ rather than $\mathbf{A}, \mathbf{C}$ in section 3.

As in [4], definite two contravariant functors:

$$\mathbf{C} \xrightarrow{F = ()^*} \mathbf{A}, \quad \mathbf{A} \xrightarrow{G = ()^o} \mathbf{C}$$

for $A \in Ob\mathbf{A}$, and $C \in Ob\mathbf{C}$, there are two natural maps η_C and ζ_A such that (see [4]):

$$\zeta_A: A \to A^{o*} = FG(A), \quad \langle \zeta_A(a), f \rangle = \langle f, a \rangle, \quad \forall f \in A^o, a \in A, \qquad (1.1)$$

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$$\eta_C: C \to C^{*o} = GF(C), \ \langle \eta_C(c), g \rangle = \langle g, c \rangle, \ \forall g \in C^*, \ c \in C. \tag{1.2}$$

We summarize some well-known results of [1-3], which will be used in the sequel.

Definition 1 *Let A be an algebra.*

(a) *A is called proper if ζ_A is injective.*

(b) *A is called weakly reflexive if ζ_A is surjective.*

(c) *A is called reflexive if ζ_A is bijective.*

Definition 2 *Let C be a coalgebra.*

(a) *C is called coreflexive if η_C is surjective.*

(b) *C is strongly coreflexive if C^* is almost noetherian.*

Remark η_C is always injective.

Lemma 1 (a) *$f \in \mathrm{Hom}_{\mathbf{C}}(C, D)$ is injective (surjective) iff $f^* \in \mathrm{Hom}_{\mathbf{A}}(D^*, C^*)$ is surjective (injective).*

(b) *If $g \in \mathrm{Hom}_{\mathbf{A}}(A, B)$ is surjective, then $g^o \in \mathrm{Hom}_{\mathbf{C}}(B^o, A^o)$ is injective.*

(c) *Let B be proper and A weakly reflexive. If $g \in \mathrm{Hom}_{\mathbf{A}}(A, B)$ is injective, then $g^o \in \mathrm{Hom}_{\mathbf{C}}(B^o, A^o)$ is surjective.*

(d) *Let A and B be reflexive. $g \in \mathrm{Hom}_{\mathbf{A}}(A, B)$ is injective (surjective) iff $g^o \in \mathrm{Hom}_{\mathbf{C}}(B^o, A^o)$ is surjective (injective).*

Proof. See [1-3].

2 Equivalentness and Its Applications

Let $\mathbf{A}_1$ be the subcategory of $\mathbf{A}$, consisting of reflexive algebras, $\mathbf{C}_1$ the subcategory of $\mathbf{C}$, consisting of coreflexive coalgebras.

Theorem 1 *$\mathbf{A}_1$ and $\mathbf{C}_1$ are a pair of equivalent category.*

Proof. We proceed in two steps.

(1) We will prove there exist natural transformation η from identity functor $1_{\mathbf{C}}$ to product functor $GF = (\ )^{*o}$:

$$1_{\mathbf{C}} \overset{\eta}{\sim} GF, \quad \eta C \mapsto \eta_C, \tag{2.1}$$

where η_C is definited by (1.2).

Also, a natural transformation ζ from $1_{\mathbf{A}}$ to $FG = (\ )^{o*}$:

$$1_{\mathbf{A}} \overset{\zeta}{\sim} FG, \quad \zeta A \mapsto \zeta_A, \tag{2.2}$$

where ζ_A is as (1.1).

It is sufficient for (2.1) to verify the commutative digramm

$$\begin{array}{ccc} C & \xrightarrow{\eta_C} & C^{*o} \\ \downarrow f & & \downarrow f^{*o}. \\ D & \xrightarrow{\eta_D} & D^{*o} \end{array}$$

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In fact, $\forall\ c \in C$, $\forall\ d \in D$, we have

$$\langle f^{*} \circ \eta_d(c), d^{*} \rangle = \langle \eta_C(c), f^{*}(d^{*}) \rangle = \langle f^{*}(d^{*}), c \rangle = \langle d^{*}, f(c) \rangle = \langle \eta_D f(c), d^{*} \rangle$$

Similarly, (2.2) is verified.

(2) Restriction $F|_{\mathbf{C}_1}$ *is a functor from* $\mathbf{C}_1$ to $\mathbf{A}_1$. i.e., $C \in \mathrm{Ob}\,\mathbf{C}_1$ implies $F(C) \in \mathrm{Ob}\,\mathbf{A}_1$. i.e., $\zeta_{F(C)}$ is an isomorphism.

Acting F on the following diagram

$$C \xrightarrow{\eta_C} GF(C),$$

we obtain

$$FG(F(C)) \xrightarrow{F(\eta_C)} F(C).$$

On the other hand, we have the inverse digramm

$$F(C) \xrightarrow{\zeta_{F(C)}} FG(F(C)),$$

and the equality

$$F(\eta_C)\zeta_{F(C)} = 1_{F(C)} \tag{2.3}$$

holds. In fact, for any $f \in F(C)$, $x \in C$, it is easy to verify the following equatities:

$$\langle F(\eta_C)\zeta_{F(C)}(f), x \rangle = \langle \zeta_{F(C)}(f), \eta_C(x) \rangle = \langle \eta_C(x), f \rangle = \langle f, x \rangle.$$

Note η_C is an isomorphism for any $C \in \mathrm{Ob}\,\mathbf{C}_1$. Furthermore, $F(\eta_C)$ is an isomorphism, and so is $\zeta_{F(C)}$ by (2.3).

Similarly, the restriction $G|_{\mathbf{A}_1}$ is a functor from $\mathbf{A}_1$ to $\mathbf{C}_1$, and equality

$$G(\zeta_A)\eta_{G(A)} = 1_{G(A)} \tag{2.4}$$

holds.

All η_C and ζ_A are isomorphisms, for any $C \in \mathrm{Ob}\,\mathbf{C}_1$, $A \in \mathrm{Ob}\,\mathbf{A}_1$, and we have by (1) and (2),

$$GF \overset{\eta}{\cong} 1_{\mathbf{C}_1} \text{(natural isomorphism)}, \tag{2.5}$$

$$FG \overset{\zeta}{\cong} 1_{\mathbf{A}_1} \text{(natural isomorphism)}. \tag{2.6}$$

This completes the proof.

Introduce some applications of theorem 1. (1) According to theorem, F is a bijection of $\mathrm{Hom}_{\mathbf{C}_1}(C, D)$ onto $\mathrm{Hom}_{\mathbf{A}_1}(D^{*}, C^{*})$, while G is a bijection of $\mathrm{Hom}_{\mathbf{A}_1}(A, B)$ *onto* $\mathrm{Hom}_{\mathbf{C}_1}(B^{o}, A^{o})$. This therefore produces interesting results: for any algebra morphism $f: D^{*} \to C^{*}$, there exists a unique coalgebra morphism $f_1: C \to D$ such that $f_1^{*} = f$; and for any coalgebra morphism $g: B^{o} \to A^{o}$, there exist a unique algebra morphism $g_1: A \to B$ such that $g_1^{o} = g$. In general, it is incorrect (see [1]). Moreover, $f \in \mathrm{Hom}_{\mathbf{C}_1}(C, D)$ is injective (surjetive) iff $F(f) = f^{*} \in \mathrm{Hom}_{\mathbf{A}_1}(D^{*}, C^{*})$ is surjective (injective), while $g \in \mathrm{Hom}_{\mathbf{A}_1}(A, B)$ is injective (surjective) iff $G(g) = g^{o} \in \mathrm{Hom}_{\mathbf{C}_1}(B^{o}, A^{o})$ is surjective (injective). This is extremly generalizes Proposition 4.1—4.3 of [1].

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(2) Recall that $(\ )^*$ and $(\ )^o$ are a pair of adjoint functors, but the proof is rather indirect, and difficult to verify (see Theorem 6.0.5 of [4]). Using (2.3) and (2.4), one prove this theorem by calculation, then obtain a concrete understanding.

3 Extensionality and Applications

In this section, we slightly modify $\mathbf{A}$ by $\widetilde{\mathbf{A}}$, $\mathbf{C}$ by $\widetilde{\mathbf{C}}$ as follows:

(1) $Ob\widetilde{\mathbf{A}} = \mathrm{Ob}\mathbf{A}$ but $\mathrm{Hom}_{\widetilde{\mathbf{A}}}(\mathbf{A}, \mathbf{B}) = \mathrm{Hom}_{\mathbf{k\text{-}mod}}(A, B) \supset \mathrm{Hom}_{\widetilde{\mathbf{A}}}(A, B)$.

(2) $Ob\widetilde{\mathbf{C}} = Ob\mathbf{C}$ but $Hom_{\widetilde{\mathbf{C}}}(C, D) = \mathrm{Hom}_{\mathbf{k\text{-}mod}}(C, D) \supset \mathrm{Hom}_{\mathbf{C}}(A, B)$.

Therefore, $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{C}}$ are full subcategories of **k- mod**, and a sequence is exact in $\widetilde{\mathbf{A}}$ or $\widetilde{\mathbf{C}}$ means so is it in **k- mod**, no any confusion between epics and surjections, etc.

In many categories, we usually study one problem such that: for an exact sequence

$$0 \to A_1 \to A \to A_2 \to 0$$

if A_1 and A_2 satisfy conditions (P), how about A? If yes, the (P) is called extensionable. For example, almost noetherian algebra and rational module are extensionable (see [2—4]). We will prove (co-) reflexivity is extensionable.

Lemma 2 (*a*) *$F = (\ )^*$ is an exact functor over $\widetilde{\mathbf{C}}$*

(*b*) *$G = (\ )^o$ is left exact functor over $\widetilde{\mathbf{A}}$.*

(*c*) *Let $0 \to A_1 \to A \to A_2 \to 0$ is an exact sequence in $\widetilde{\mathbf{A}}$.* If A_1 is weakly reflexive and A proper, then

$$0 \to A_2^o \to A^o \to A_1^o \to 0$$

also exact.

Proof. For any linear map h: $V_1 \to V_2$ where V_i is vector space, the equalities

$$\mathrm{Ker}h^* = (\mathrm{Im}h)^{\perp} \qquad \mathrm{Im}h^* = (\mathrm{Ker}h)^{\perp}$$

hold. Hence, (a) is valid. (c) holds by (b) and by Lemma 1 (c). We prove (b) in two steps.

(1) Let $0 \to A_1 \xrightarrow{f} A \xrightarrow{g} A_2 \to 0$ be exact. In order prove left exact sequence

$$0 \to A_2^* \xrightarrow{g^o} A^o \xrightarrow{f^o} A_1^o$$

it is enough to verify $\mathrm{Ker} f^o = \mathrm{Im}; g^o$. Note f^o is restriction of f^* over A^o, then

$$\mathrm{Ker}f^o = \mathrm{Ker}; f^* \cap A^o = (\mathrm{Im}f)^{\perp} \cap A^o.$$

(2) Since g^o is the restrition of g^* over A_2^o, then

$$\mathrm{Im}g^o = g^*(A_2^*) = g^*(A_2^* \cap A_2^o) \subset \mathrm{Im}g^* \cap A^o. \tag{3.1}$$

On the other hand, for any $a^o \in \mathrm{Im}g^* \cap A^o$, there exists $b^* \in A_2^*$ satisfying

$$g^*(b^*) = a^o, \ J \subset \mathrm{Ker}a^o \subset A,$$

where J is a cofinite idea of A. It is easy to see

$$g(J) \subset \mathrm{Ker} b^{*}.$$

Since g is surjection, $g(J)$ is also cofinite idea of A_2. Hence, $b^{*} \in A_2^{o}$, i.e.

$$g^{*}(A_2^{*}) \cap A^{o} \subset g^{o}(A_2^{o}) = \mathrm{Im} g^{o}. \tag{3.2}$$

By (3.1)-(3.2), the equalities

$$\mathrm{Im} g^{o} = \mathrm{Im} g^{*} \cap A^{o} = (\mathrm{Ker}; g)^{\perp} \cap A^{o}.$$

Finally, we reach the following equalitis

$$\mathrm{Ker} f^{o} = \mathrm{Ker} f^{*} \cap A^{o} = (\mathrm{Im} f)^{\perp} \cap A^{o} = (\mathrm{Ker} g)^{\perp} \cap A^{o} = \mathrm{Im} g^{o}.$$

Theorem 2 *Let* $0 \to C_1 \to C \to C_2 \to 0$ *be an exact sequence in* $\widetilde{\mathbf{C}}$

(*a*) *If* C_1 *and* C_2 *are strongly coreflexive, then so is* C.

(*b*) *Suppose* C_2 *is coreflexive, then* C_1 *coreflexive* $\Leftrightarrow C$ *coreflexive. In particular, coreflexivity is extensionable.*

Proof (a) Since $(\ )^{*}$ is an exact functor, then

$$0 \to C_2^{*} \to C^{*} \to C_1^{*} \to 0 \tag{3.3}$$

also exact. (a) holds due to extensionality of almost noetherian algebras.

(b) In (3.3), C_2^{*} is reflexive by Theorem 1, and C_1^{*} is always proper (see[4] 6.0.3).

Using Lemma 2 (c) and (2.1), we obtain the following commutative diagam:

$$\begin{array}{ccccccccc}
0 & \to & C_1 & \to & C & \to & C_2 & \to & 0 \\
 & & \downarrow \eta_{C_1} & & \downarrow \eta_C & & \downarrow \eta_{C_2} & & \\
0 & \to & C_1^{*o} & \to & C^{*o} & \to & C_2^{*o} & \to & 0
\end{array}.$$

Note η_{C_2} is an isomorphism. η_C isomorphism $\Leftrightarrow \eta_{C_1}$ isomorphism by Short Five Lemma, i.e. C coreflexive $\Leftrightarrow C_1$ coreflexive.

This completes the proof.

In dual, we have another theorem.

Theorem 3 *Let exact sequence*

$$0 \to A_1 \to A \to A_2 \to 0$$

keeps exactness under action of $G = (\ )^{o}$.

(*a*) *If* A_1 *and* A_2 *are proper (weakly reflexive), then so is* A.

(*b*) *If any two of* A_1, A_2 *and* A *are reflexive, then so is the third.*

In particular, (weakly) reflexivity is extensionable.

This proof is dual to the proof above, so we omit it.

Theorem 2—3 can be used to generalize and simplify some results of [1- 3]. Some propositions in [1- 3], which are rather difficult proven, become obvious here. We cite examples, (1). If C_1 and C_2 are (strongly) coreflexive, then so is $C_1 \oplus C_2$. (2). If A_1 and A_2 are proper (weakly) reflexive, then so is $A_1 \oplus A_2$. Due to the shortage of space we don't give examples any more.

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关于(余)反射的范畴性质及其应用

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摘 要

本文研究了(余)反射的范畴性质及应用,得到的主要结果如下:(1) 余反射余代数范畴和反射代数范畴是等价的 (2) 反射和余反射都具有可扩张性 使用这些结果可以用来推广和化简了[1—4]**等人的部分工作**