Journal of Mathematical Research & Exposition
Vol. 18, No. 1, 30-32, February 1998

From Optimal Stopping Problems over Tree Sets to Optimal Stopping Problems over Partially Ordered Sets*

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Abstract In this paper, we discuss relations between optimal stopping problems over tree sets and partially ordered sets, prove that there is a 1-1 correspondence between them and so every optimal strategy can be obtained in the set of optimal control variables.

Keywords partially ordered set, optimal stopping, optimal strategy.

Classification AMS(1991) 64G40/O211.4

In 1966, Haggstrom first introduced control variable to discuss optimal stopping problems over tree sets.[1] The study of this problem had remained at a standstill untill 1983 G. F. Lawler and R. J. Vanderbei discussed optimal stopping problems over partially ordered sets using the theory of multi-parameter stochastic processes.[2] Afterwards, in 1994, Writer proved the equivalence of optimal strategy and admissible strategy.[3] In this paper, we discuss relations between optimal stopping problems over tree sets and partially ordered sets and proved that every optimal strategy can be obtained in the set of optimal control variables.

Let $(\Omega, \mathbf{T}, P)$ be a complete probability space, $N = \{0, 1, 2, \cdots\}$, and $(S, \preceq)$ be a partially ordered set as in [2]. Let $Z_{s,u}, u \in U(s), s \in S$, and $Z_s, s \in S$, be integrable random variables with A^+ conditions[3] such that $Z_{s,u} \in \mathbf{F}_u, Z_s \in \mathbf{F}_s$, where $U(s)$ is the set of direct successors of s. Z_s is called the stopping reward at s and $Z_{s,u}$ the running reward from s to u. $\mathbf{F}_s, s \in S\}$ is the increasing family of sub-σ-algeberas of $\mathbf{T}$ generated by reward processes.

Let $A = \{(a_0, a_1, \cdots, a_j): a_0 = 0,\ a_i \in S,\ a_{i+1} \in U(a_i),\ 0 \le i \le j, j \in N\}$. A partial order $\preceq$ is defined on A as follows: If $a = (a_0, a_1, \cdots, a_j)$ and $b = (b_0, b_1, \cdots, b_k)$ are elements of A, then $a \preceq b$ if and only if $j \le k$ and $a_i = b_i$ for all $i \le j$. It follows that $(A, \preceq)$ is a tree set. For any $a = (a_0, a_1, \cdots, a_j) \in A$, Z_a is defined by $Z_a = \sum_{i=0}^{j-1} \alpha^i Z_{a_i, a_{i+1}} + \alpha^j Z_{a_j}$, where $\alpha \in (0, 1)$ is called the discount factor. Let $\mathbf{T}_a = \sigma\{Z_b: b \preceq a,\ b \in A\}$. The ideas and notations in the following theorem can be referred to [1], [2], [3]. The set of all control variables on $(A, \preceq)$ is denoted by $\sum$ and the set of all strategies on $(S, \preceq)$ by $\mathbf{T}$.

Theorem *There is a 1-1 correspondence φ between $\sum$ and $\mathbf{T}$*, and $EZ_t = EZ(\varphi(t))$ for every $t \in \sum$.

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* Received August 30, 1995. Supported by the National Science Foundation of China.

Corollary *Under the same condition of the theorem, for given optimal strategy $T^* \in \mathbf{T}$, there exists an optimal control variable $t^* \in \sum$ such that $EZ_{t^*} = EZ(T^*)$.*

Proof of Theorem For any $t \in \sum$, we define a correspondence φ on $\sum$ as follows:
$(\sigma_0, \sigma_1, \cdots, \sigma_j)(\omega) = (a_0, a_1, \cdots, a_j)$ and $\sigma_k(\omega) = \sigma_j(\omega)$, $k > j$ on $\{t = a = (a_0, a_1, \cdots, a_j)\}$

$$\tau = \inf\{k: \sigma_k(\omega) = \sigma_{k+1}(\omega)\}.$$

We get $\varphi(t) = T = ((\sigma_k), \tau)$. First, we prove that $\varphi(t) \in \mathbf{T}$. For $n \in N$ and $s \in S$, we have

$$\{\sigma_n = s\} = \bigcup_{b_0, b_1, \cdots, b_{n-1}} \{t \succeq (b_0, b_1, \cdots, b_{n-1}, s)\} \in \mathbf{F}_s$$

Hence, σ_n is a stopping point on $(S, \preceq)$ and it is obvious that $\sigma_n \preceq \sigma_{n+1}$. To show that $\sigma_{n+1} \in \mathbf{F}_{\sigma_n}$, it is needed to prove that for any u in S, we have $\{s_{n+1} = u\} \cap \{\sigma_n = s\} \in \mathbf{F}_s$ for each s in S.

If $u = s$, then

$$\{\sigma_{n+1} = u\} \cap \{\sigma_n = s\} = \bigcup_{a_0, a_1, \cdots, a_{n-1}} \{t = (a_0, a_1, \cdots, a_{n-1}, s\} \in \mathbf{F}_s$$

If $u > s$, then $u \in U(s) = \{s^1, s^2, \cdots, s^q\}$ and

$$\{\sigma_{n+1} = s^i\} \cap \{\sigma_n = s\} = \bigcup_{a_0, a_1, \cdots, a_{n-1}} \{t \succeq (a_0, a_1, \cdots, a_{n-1}, s, s^i)\}$$

$$= \bigcup_{a = (a_0, a_1, \cdots, a_{n-1}, s), a_0, a_1, \cdots, a_{n-1}} \{t \succeq ai\} \in \mathbf{F}_s,$$

where $ai = (a_0, a_1, \cdots, a_{n-1}, s, s^i)$, $1 \leq i \leq q$. It follows that $T = ((\sigma_n), \tau) \in \mathbf{T}$ and clearly φ is a single correspondence.

Now we prove that for any $T \in \mathbf{T}$ there is a $t \in \mathbf{T}$ such that $\varphi(t) = T$. Suppose that $T = ((\sigma_n), \tau)$ is a given strategy. Let

$$B_0 = \{\tau = j\} \cap \{(\sigma_0, \sigma_1, \cdots, \sigma_j) = a = (a_0, a_1, \cdots, a_j)\}$$

$$B_k = \{(\sigma_0, \sigma_1, \cdots, \sigma_j, \sigma_{j+1}) = ak\}, ak = (a_0, \cdots, a_j, a_j^k), a_j^k \in U(a_j) = \{a_j^1, \cdots, a_j^q\}$$

$$B_{q+1} = \Omega \setminus \bigcup_{i=0}^{q} B_i,$$

then $B_i \in \mathbf{F}_{a_j}$, $i = 0, 1, \cdots, q+1$, $B_i \cap B_j = \emptyset$, $i \neq j$ and $\bigcup_{i=0}^{q+1} B_i = \Omega$. In the view of actualities, the information before the point which the known path has reached is conditionally independent of the information contained in the known path. We have

$$P\{B_i | \mathbf{F}_a\} P\{B_j | \mathbf{F}_a\} = P\{B_i B_j | \mathbf{F}_a\} = 0, \quad i \neq j.$$

Let $C_j = \{P\{B_j | \mathbf{F}_n\} > 0\}$, $0 \leq i \leq q+1$ then $C_i \cap C_j = \emptyset$ a. s $i \neq j$. Let $m_i = P\{B_i | \mathbf{F}_a\}$, then $m_i \geq 0$ and $\sum_{i=0}^{q+1} m_i = 1$, so that $m_i^2 = m_i$. Hence, $m_i = I\{C_i\}$. It shows that $\{C_i, 0 \leq i \leq q+1\}$ is $\mathbf{F}_a$- measured cutting to Ω and $B_i = C_i$ a. s By completeness, B_i is $\mathbf{F}_a$- measured for $i = 0, 1,$

$\cdots, q+1$. Now we define $t(\omega)=a=(a_0, a_1, \cdots, a_j)$ on B_0, then $\{t=a\}=B_0 \in \mathbf{F}_a$ and $\{t \succeq a\}=B_k \in \mathbf{F}_a$, $k=1, 2, \cdots, q$. From $P\{\tau<\infty\}=1$, we know $P\{t \in A\}=1$. Hence $t \in \sum$ and it is clear that $\varphi(t)=T$ and $EZ_t=E\varphi(t)=ET$.

References

[1] G. W. Haggstrom, Ann. Math. Sat., **37**(1966), 7—60.

[2] G. F. Lawler and R. J. Vanderbei, Ann. Probab., **11**(1983), 642—647.

[3] Yi Dongyun, Chinese J. Appl. Probab. Statist., (1994), 6—11.

[4] Yi Dongyun, J. Math. Res. Exp. (Chinese), **4**(1993), 619—622.

从树型集上最优停止问题到一般偏序集上最优停止问题

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摘　要

本文讨论了树型集上与偏序集上最优停止问题两者间的关系, 证明了最优策略与最优控制变量的一一对应关系, 从而导出最优策略 可在最优控制变量中取到