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Some Combinatorial Identities on Lattice-Point Poset*

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Abstract This paper is devoted to the counting of j-chains of lattice-point poset. The incidence function and two basic enumerators associated with it are introduced and evaluated. Our results contain a lot of combinatorial sums.

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1. Introduction

The following formulae are widely used combinatorial identities[1].

$$\sum_{k=0}^{p}\binom{n}{k}\binom{m}{p-k}=\binom{n+m}{p}, \tag{1.1}$$

$$\sum_{k=0}^{n}\binom{k+x}{x}\binom{n-k+y}{y}=\binom{n+x+y+1}{x+y+1}, \tag{1.2}$$

$$\sum_{k=0}^{n}\frac{\alpha}{ak+\alpha}\binom{ak+\alpha}{k}\frac{\beta}{a(n-k)+\beta}\binom{a(n-k)+\beta}{n-k}$$
$$=\frac{\alpha+\beta}{an+\alpha+\beta}\binom{an+\alpha+\beta}{n}. \tag{1.3}$$

All these formulae have their respective combinatorial proofs. For instance, in order to show (1.1), we only need to see that the right-hand side of (1.1) is the number of p-element subsets X of $[m+n]=\{1,2,\cdots,m+n\}$. Suppose that X intersects $[n]$ in k elements. There are $\binom{n}{k}$ choices for $X\cap[n]$, and $\binom{m}{p-k}$ choices for the remaining $p-k$ elements $X\cap\{n+1,n+2,\ldots,n+m\}$. Thus there are $\binom{n}{k}\binom{m}{p-k}$ ways that $X\cap[m+n]$ can have p elements, and summing over k gives the total number $\binom{m+n}{p}$ of p-element subsets of $[m+n]$. Take set of

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lattice-paths instead of $[m+n]$ in the above argument, we get combinatorial proofs of (1.2) and (1.3) at once. See [2, 3] for details. In the present paper, we apply this idea to the lattice-point poset (partially ordered set, for short), and obtain some combinatorial identities which may be viewed as generalizations of (1.1)-(1.3). For simplicity, the reader is referred to [5] for the notations and terminology of poset.

2 Main Results

Let N^m be the set of all m-tuples $\overline{x}=(x_1, x_2, \cdots, x_m)$ of non-negative integers, called lattice point, and let $\overline{n}=(n_1, n_2, \cdots, n_m)$ be a fixed element of N^m. We may make N^m be a poset with order relation $\leq$ defined by $\overline{x}\leq\overline{y}$ iff $x_i\leq y_i$, $i=1,2,\cdots,m$.

Definition 1 *A lattice-point poset with respect to $\overline{n}$ is the principal order ideal generated by $\overline{n}$, namely say, $\Lambda_m(\overline{n})=\{\overline{x}\in N^m \mid \overline{x}\leq\overline{n}\}$. A linearly ordered j-subset of $\Lambda_m(\overline{n})$ is called a j-chain of $\Lambda_m(\overline{n})$.*

The purpose of this paper is to count j-chains of $\Lambda_m(\overline{n})$. We begin with some elementary enumerative functions as follows.

Definition 2 *Suppose that $A_m{}^j(\overline{n})$ is the set of all j-chains of $\Lambda_m(\overline{n})$ and let $B_m{}^j(\overline{n})$ and $C_m{}^j(\overline{n})$ denote the sets of the j-chains in $\Lambda_m(\overline{n})$ that contain $\overline{0}$ and $\overline{0}$ and $\overline{n}$, respectively. Then, G-, F-and λ-function are respectively defined by $G_m^j(\overline{n})=|A_m{}^j(\overline{n})|$, $F_m^j(\overline{n})=|B_m{}^j(\overline{n})|$ and $\lambda_m^j(\overline{n})=|C_m{}^j(\overline{n})|$. λ-function is also called the incidence function of $\Lambda_m(\overline{n})$.*

In what follows, we adopt some multidimensional notations, only for convenience. $\forall\ \overline{n}=(n_1, n_2, \cdots, n_m)\in N^m$, define $\Delta(\overline{n})=(n_2, n_3, \cdots, n_m)$ and $|\overline{n}|=\sum_{i=0}^{m} n_i$. Let $H(\overline{n})$ be a function on N^m. Then we write $\overline{n}=(n_1, \Delta(\overline{n}))$ and $H(\overline{n})=H(n_1, \Delta(\overline{n}))$.

Here, we come to our main results.

Theorem 1 *Let $\Lambda_m(\overline{n})$ be defined as above. Then*

$$F_m^j(\overline{n})=F_m^j(n_1-1, \Delta(\overline{n}))+\sum_{\Delta(\overline{x})\leq\Delta(\overline{n})}\sum_{j_1+j_2=j}F_{m-1}^{j_1}(\Delta(\overline{n}-\overline{x}))F_m^{j_2}(n_1-1, \Delta(\overline{x})), \tag{2.1}$$

$$\lambda_m^j(\overline{n})=F_m^{j-1}(n_1-1, \Delta(\overline{n}))+\sum_{\overline{0}<\Delta(\overline{x})\leq\Delta(\overline{n})}\sum_{j_1+j_2=j}\lambda_{m-1}^{j_1}(\Delta(\overline{n}-\overline{x}))F_m^{j_2}(n_1-1, \Delta(\overline{x})), \tag{2.2}$$

$$G_m^j(\overline{n})=F_m^j(\overline{n})+F_m^{j+1}(\overline{n}), \tag{2.3}$$

$$F_m^j(\overline{n})=\lambda_m^j(\overline{n})+\lambda_m^{j+1}(\overline{n}), \tag{2.4}$$

where $\overline{n}-\overline{x}=(n_1-x_1, n_2-x_2, \cdots, n_m-x_m)$.

Proof We only prove (2.1), the rest can be proved in the similar way. To do this, observe that the left-hand side of (2.1) is just the cardinality of $B_m{}^j(\overline{n})$. Let X be an arbitrary element of $B_m{}^j(\overline{n})$. Obviously, X may intersect the hyperplane $x_1=n_1$ or not. If X does not intersect the hyperplane $x_1=n_1$, then there are $F_m^j(n_1-1, \Delta(\overline{x}))$ choices for $X\in B_m^j(\overline{n})$. Suppose that X intersects the hyperplane $x_1=n_1$ in j_1 lattice points which contain $\overline{x}$ as the minimal lattice point. There are $F_{m-1}^{j_1}(\Delta(\overline{n}-\overline{x}))$ choices for $X\cap\{\overline{y}\mid\overline{x}\leq\overline{y}\leq\overline{n}\}$, and $F_m^{j_2}(n_1-1, \Delta(\overline{x}))$ $(j_1+j_2=j)$

choices for the remaining j_2 lattice points $X \cap \{\overline{y} \mid \overline{y} \le \overline{x},\ y_1 \le n_1 - 1\}$. Thus there are $F^{j_1}_{m-1}(\Delta(\overline{n} - \overline{x}))F^{j_2}_m(n_1 - 1, \Delta(\overline{x}))$ ways totally that $X \cap \{\overline{y} \mid \overline{x} \le \overline{y} \le \overline{n}\}$ can have j_1 lattice points of which $\overline{x}$ is the minimal lattice point. Summing over $\Delta(\overline{x})$ and $\{(j_1, j_2) \mid j_1 + j_2 = j\}$ gives the total number of j-chains X that intersects the hyperplane $x_1 = n_1$. This completes the proof. □

Certainly, these recurrence relations will lead to the explicit forms of F-, G- and λ-functions.

Theorem 2 *Let $\Lambda_m(\overline{n})$ be the lattice-point poset with respect to $\overline{n}$, and F-, G- and λ-function as given by Definition 2. Then*

$$F^h_m(\overline{n}) = \sum_{\nabla(\overline{x}) \le \nabla(\overline{n})} \begin{pmatrix} \nabla(\overline{n}) \\ \nabla(\overline{k}) \end{pmatrix} \begin{pmatrix} \Delta(\overline{n}) \\ \nabla(\overline{k}) \end{pmatrix}_0 \tag{2.5}$$

$$G^h_m(\overline{n}) = \sum_{\nabla(\overline{x}) \le \nabla(\overline{n})} \begin{pmatrix} \nabla(\overline{n}) \\ \nabla(\overline{k}) \end{pmatrix} \begin{pmatrix} \Delta(\overline{n}) \\ \nabla(\overline{k}) \end{pmatrix}_1 \tag{2.6}$$

$$\lambda^h_m(\overline{n}) = \sum_{\nabla(\overline{x}) \le \nabla(\overline{n})} \begin{pmatrix} \nabla(\overline{n}) \\ \nabla(\overline{k}) \end{pmatrix} \begin{pmatrix} \Delta(\overline{n}) \\ \nabla(\overline{k}) \end{pmatrix}_{-1}, \tag{2.7}$$

where $\nabla(\overline{k}) = (k_1, k_2, \cdots, k_{m-1})$, $\begin{pmatrix} \nabla(\overline{n}) \\ \nabla(\overline{k}) \end{pmatrix} = \prod_{i=1}^{m-1} \begin{pmatrix} n_i \\ m_i \end{pmatrix}$, *and*

$$\begin{pmatrix} \Delta(\overline{n}) \\ \nabla(\overline{k}) \end{pmatrix}_s = \begin{pmatrix} n_1 \\ i \end{pmatrix} \begin{pmatrix} n_2 + i \\ i \end{pmatrix} \begin{pmatrix} n_2 \\ j \end{pmatrix} \begin{pmatrix} n_3 + i + j \\ i + j \end{pmatrix} \cdots$$
$$\begin{pmatrix} n_{m-1} \\ k \end{pmatrix} \begin{pmatrix} n_m + i + j + \cdots + k \\ i + j + \cdots + k \end{pmatrix} \begin{pmatrix} n_m \\ h + s - 1 - (i + j + \cdots + k) \end{pmatrix}.$$

Proof We only prove (2.5). (2.7) can be proved by the same argument and (2.6) can be verified directly by (2.3) and (2.5). We first apply induction on m. It is clear that (2.5) holds for $m = 3D\,1$. Suppose that $m > 1$ and (2.5) holds for $m' = m - 1$. Let $\overline{n} = (n_1, n_2, \cdots, n_m) \in N^m$. We proceed induction on n_1. When $n_1 = 0$, it follows from the definition of F-function that $F^h_m(\overline{n}) = F^h_{m-1}(\Delta(\overline{n}))$. From the above hypothesis, we have

$$F^h_{m-1}(\Delta(\overline{n})) = \sum_{i=0}^{n_2} \begin{pmatrix} n_2 \\ i \end{pmatrix} \begin{pmatrix} n_3 + i \\ i \end{pmatrix} \sum_{j=0}^{n_3} \begin{pmatrix} n_3 \\ j \end{pmatrix} \begin{pmatrix} n_4 + i + j \\ i + j \end{pmatrix} \sum \cdots \sum_{k=0}^{n_{m-1}}$$
$$\begin{pmatrix} n_{m-1} \\ k \end{pmatrix} \begin{pmatrix} n_m + i + j + \cdots + k \\ i + j + \cdots + k \end{pmatrix} \begin{pmatrix} n_m \\ h - 1 - (i + j + \cdots + k) \end{pmatrix}.$$

This implies (2.5) holds for $n_1 = 0$. We further assume that $n_1 > 0$ and (2.5) holds for any= 20 $\overline{n}' = (n'_1, n'_2, \cdots, n'_m) \in N^m$ with $n'_1 \le n_1 - 1$. Then from both hypotheses, we may get by (2.1) that

$$F_m^h(\overline{n}) = \sum_{i=0}^{n_1-1}\binom{n_1-1}{i}\binom{n_2+i}{i}\sum_{j=0}^{n_2}\binom{n_2}{j}\binom{n_3+i+j}{i+j}\sum\cdots$$
$$\sum_{k=0}^{n_{m-1}}\binom{n_{m-1}}{k}\binom{n_m+i+j+\cdots+k}{i+j+\cdots+k}\binom{n_m}{h-1-(i+j+\cdots+k)}$$
$$+\sum_{i=0}^{n_1-1}\binom{n_1-1}{i}\binom{n_2+1+i}{1+i}\sum_{j=0}^{n_2}\binom{n_2}{j}\binom{n_3+1+i+j}{1+i+j}\sum\cdots$$
$$\sum_{k=0}^{n_{m-1}}\binom{n_{m-1}}{k}\binom{n_m+1+i+\cdots+k}{1+i+j+\cdots+k}\binom{n_m}{h-2-(i+j+\cdots+k)}$$
$$=\sum_{i=0}^{n_1}\left\{\binom{n_1-1}{i}+\binom{n_1-1}{i-1}\right\}\binom{n_2+i}{i}\sum_{j=0}^{n_2}\binom{n_2}{j}\binom{n_3+i+j}{+i+j}\sum\cdots$$
$$\sum_{k=0}^{n_{m-1}}\binom{n_{m-1}}{k}\binom{n_m+i+j+\cdots+k}{i+j+\cdots+k}\binom{n_m}{h-1-(i+j+\cdots+k)}$$
$$=\sum_{i=0}^{n_1}\binom{n_1}{i}\binom{n_2+i}{i}\sum_{j=0}^{n_2}\binom{n_2}{j}\binom{n_3+i+j}{i+j}\sum\cdots$$
$$\sum_{k=0}^{n_{m-1}}\binom{n_{m-1}}{k}\binom{n_m+i+j+\cdots+k}{i+j+\cdots+k}\binom{n_m}{h-1-(i+j+\cdots+k)}$$

as desired. $\square$

Theorem 3 *Let* $\overline{n}\in N^m$. *Then*

$$\prod_{l=1}^{m}\binom{n_l+j}{j} = \sum_{i=0}^{j}\binom{j}{i}F_m^{i+1}(\overline{n}), \qquad or \tag{2.8}$$

$$F_m^{j+1}(\overline{n}) = \sum_{i=0}^{j}(-1)^{j-i}\binom{j}{i}\prod_{l=1}^{m}\binom{n_l+i}{i}. \tag{2.9}$$

Note that (2.8) *holds for* $j\le\lceil\overline{n}\rceil$.

Proof Clearly, when $j\le\lceil\overline{n}\rceil$, (2.8) and (2.9) is a pair of inverse relation, it suffices to prove (2.8). Let $D(\overline{n}, j) = \{(\overline{a_1}, \overline{a_2}, \cdots, \overline{a_j})\,|\,0<\overline{a_1}\le\overline{a_2}\le\cdots\le\overline{a_j}, \overline{a_i} = (n_i^{(1)}, n_i^{(2)}, \cdots, n_i^{(m)})\in\Lambda_m(\overline{n})\}$ and $D(\overline{n}, j;\ i) = \{(\overline{a_1}, \overline{a_2}, \cdots, \overline{a_j})\,|\,0<\overline{a_1}\le\overline{a_2}\le\cdots\le\overline{a_j}$ with exactly i distinct lattice points in $\overline{a_1}$, $\overline{a_2}, \cdots, \overline{a_j}\}$, $i = 0, 1, \cdots, j$, $j\le\lceil\overline{n}\rceil$. It is easy to see that $D(\overline{n}, j) = \bigcup_{i=0}^{j} D(\overline{n}, j;\ i)$ and $D(\overline{n}, j; i)\cap D(n, j; i') = \varnothing$ $(i\ne i')$. Furthermore, with aid of counting of lattice paths, we know that $|D(\overline{n}, j)| = \prod_{l=1}^{m}\binom{n_l+j}{j}$. On the other hand, from the definition of $F_m^j(\overline{n})$, it follows that there are $F_m^{i+1}(\overline{n})$ choices for i distinct nonzero lattice points, without loss of generality, denoted by $\overline{a_1} < \overline{a_2} < \cdots < \overline{a_i}$, and $\binom{j}{i}$ ways such that $(\overline{a_1}, \cdots, \overline{a_1}, \overline{a_2}, \cdots, \overline{a_2}, \cdots, \overline{a_i}, \cdots, \overline{a_i}) \in D(\overline{n}, j; i)$. Thus $|D(\overline{n}, j; i)| = \binom{j}{i} F_m^{i+1}(\overline{n})$. Summing over i gives the cardinality of $D(\overline{n}, j)$. This completes the proof. □

Suppose that σ be an element of permutation group S_n of 1, 2, $\cdots$, n. Define $\sigma(\overline{n}) = (n_{\sigma(1)}, n_{\sigma(2)}, \cdots, n_{\sigma(n)})$. Based on Theorem 3, we have that $F_m^{j+1}(\overline{n}) = F_m^{j+1}(\sigma(\overline{n}))$. For $m = 2$. Then it implies that

$$\sum_{i=0}^{n}\binom{n}{i}\binom{m+i}{i}\binom{m+1}{n-i} = 3D\sum_{i=0}^{m}\binom{m}{i}\binom{n+i}{i}\binom{n+1}{n-i}. \tag{2.10}$$

Further, let $n \to \infty$, it leads to

$$\sum_{i=0}^{m}\binom{m}{i}\binom{j}{i} = \binom{m+j}{j}. \tag{2.11}$$

Remark (2.10) has been posed as an interesting problem to SIAM Review and published on it .[6] Now, many a solutions including WZ- method as well as hypergeometric series has been found.

Put $j = \lceil n \rceil$ and $\lceil n \rceil - 1$ in (2.9), respectively, we have

$$\binom{\lceil n \rceil}{n_1, n_2, \cdots, n_m} = \sum_{i=0}^{\lceil n \rceil}(-1)^{\lceil n \rceil - i}\binom{\lceil n \rceil}{i}\prod_{l=1}^{m}\binom{n_l+i}{i}, \tag{2.12}$$

$$\sum_{i=0}^{\lceil n \rceil - 1}(-1)^{\lceil n \rceil - 1}\binom{|=^{-} n| - 1}{i}\prod_{l=0}^{m}\binom{n_l+i}{i} = \frac{\lceil n \rceil^2 - \sum_{i<j} n_i n_j}{\lceil n \rceil}\binom{\lceil n \rceil}{n_1, n_2, \cdots, n_m}. \tag{2.13}$$

Also, put $m = 2$ and $n_2 = 0$ in (2.9), we have

$$\binom{n}{j} = \sum_{i=0}^{j}(-1)^{j-i}\binom{j}{i}\binom{n+i}{i}. \tag{2.14}$$

Similarly, Vondermonde- type convolution identities stated below can also be established by counting of j - chains of lattice- point poset.

Theorem 4 *Let F^-, G^- and λ-function be given by Definition 2. Then the following identities hold for all $j_1, j_2 \in N$, $\overline{n} \in N^m$.*

$$F_m^{j_1+j_2-1}(\overline{n}) = \sum_{\overline{0} \le \overline{k} \le \overline{n}} \lambda_m^{j_1}(\overline{k}) F_m^{j_2}(\overline{n} - \overline{k}), \tag{2.15}$$

$$G_m^{j_1+j_2-1}(\overline{n}) = \sum_{\overline{0} \le \overline{k} \le \overline{n}} F_m^{j_1}(\overline{k}) F_m^{j_2}(\overline{n} - \overline{k}), \tag{2.16}$$

$$\lambda_m^{j_1+j_2-1}(\overline{n}) = \sum_{\overline{0} \le \overline{k} \le \overline{n}} \lambda_m^{j_1}(\overline{k})\lambda_m^{j_2}(\overline{n}-\overline{k}). \tag{2.17}$$

Proof Suppose that $\overline{k}$ is an arbitrary element of $\Lambda_m(\overline{n})$ such that (j_1+j_2-1)-chains X with $\overline{k} \in X$ of $\Lambda_m(\overline{n})$ intersects $\Lambda_m(\overline{k})$ in j_1 lattice points and intersects $\{\overline{x} \mid \overline{k} \le \overline{x} \le \overline{n}\}$ which is isomorphic to $\Lambda_m(\overline{n}-\overline{k})$ in j_2 lattice points. Then (2.15) emerges immediately from the counting of $B_m^{j_1+j_2-1}(\overline{n})$ over all such $\overline{k}$. (2.16) and (2.17) can be derived in similar way. □

Finally, we list a few useful forms of these identities to claim their broad range.

Put $m=1$ in (2.16), we have

$$\sum_{k=1}^{n}\binom{k-1}{j_1-2}\binom{n-1-k}{j_2-2} = \binom{n-1}{j_1+j_2-3}. \tag{2.18}$$

Put $m=2$ in (2.15), we have

$$\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\left\{\sum_{i=0}^{k_1}\binom{k_1}{i}\binom{k_2+i}{i}\binom{k_2}{j_1-1-i}\right\} \times \left\{\sum_{i=0}^{n_1-k_1}\binom{n_1-k_1}{i}\binom{n_2-k_2+i}{i}\binom{n_2-k_2}{j_2-1-i}\right\} = \sum_{i=0}^{n_1}\binom{n_1}{i}\binom{n_2+i}{i}\binom{n_2+1}{j_1+j_2-1-i}. \tag{2.19}$$

Take (2.9) into (2.19). Then, we have

$$\sum_{i=0}^{j_1+j_2-2}\binom{j_1+j_2-2}{i}\binom{n_1+1+i}{1+i}\binom{n_2+1+i}{1+i} = \sum_{i=0}^{n_1}\binom{n_1}{i}\binom{n_2+i}{i}\binom{n_2+1}{j_1+j_2-1-i}. \tag{2.20}$$

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