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局部k-一致凸空间的对偶空间*

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摘 要 本文证明了若Banach空间X是局部k-一致凸的, 则对每个$x\in S(X)$, $f\in\Sigma(x)$是X^*的k-强光滑点, 并得到局部k-一致凸空间的几个性质

关键词 局部k-一致凸, k-强光滑, k-维体积
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文[1]利用k-维体积, 引进Banach空间的k-一致凸性与局部k-一致凸性; 文[2]研究了Banach空间的k-光滑性与k-强光滑性; 它们分别是一致凸性、局部一致凸性、光滑性、强光滑性概念的推广. 建立局部k-一致凸性与k-强光滑性之间的一种关系

本文总假设X是Banach空间, 记$S(X)=\{x\in X;\ \|x\|=1\}$; $U(X)=\{x\in X;\ \|x\|\le 1\}$; 对$x\in S(X)$, $\Sigma(x)=\{f\in S(X^*);\ f(x)=1\}$; 由$X$中的点$x_1, x_2, \cdots, x_{k+1}$所围成的$k$-维体积记为

$$A(x_1, x_2, \cdots, x_{k+1})=\sup\left\{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ f_1(x_1) & f_1(x_2) & \cdots & f_1(x_{k+1}) \\ \cdots & \cdots & \cdots & \cdots \\ f_k(x_1) & f_k(x_2) & \cdots & f_k(x_{k+1}) \end{vmatrix};\ f_i\in U(X^*),\ i=1, 2, \cdots, k\right\}$$

并在典型嵌入映射J下, $x\in X$与$J(x)\in X^{**}$不加区别

定义1[2] 称$x\in S(X)$为X的k-强光滑点, 是指

Ⅰ x是X的k-光滑点, 即$\dim\Sigma(x)\le k$, 其中dim表示集合的线性维数

Ⅱ 对任何点列$\{f^n\}\subset S(X^*)$, 当$f^n(x)\to 1$时, $\{f^n\}$相对列紧

如果$S(X)$的每个点都是X的k-强光滑点, 称X为k-强光滑空间

引理2 设$x\in S(X)$, 则下列命题等价

(1) 对任何$\{f_i^n\}\subset S(X^*)$, $i=1, 2, \cdots, k+1$, 当$f_i^n(x)\to 1$时, $A(f_1^n, f_2^n, \cdots, f_{k+1}^n)\to 0$.

(2) 对任何$\epsilon>0$, 存在$\delta>0$, 当$f_1, f_2, \cdots, f_{k+1}\in S(X^*)$, $(f_1+f_2+\cdots+f_{k+1})(x)>(k+1)-\delta$时, $A(f_1, f_2, \cdots, f_{k+1})<\epsilon$

(3) x是X的k-强光滑点

证明 (1)与(2)等价性的推导是平凡的, 这里从略. 下面证明(1)与(3)等价.

设x是X的k-强光滑点, 如果(1)不成立, 则存在$\{f_i^n\}\subset S(X^*)$, $f_i^n(x)\to 1$, $i=1, 2, \cdots, k+1$, 及$\epsilon_0>0$和$\{n\}$的子列(仍记作$\{n\}$), 使

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$$A(f_1^n, f_2^n, \cdots, f_{k+1}^n) \geq \epsilon_0 \tag{a}$$

由定义1的条件Ⅱ知, 存在$\{n\}$的子列$\{m\}$与$f_i \in S(X^*)$, $i = 1, 2, \cdots, k+1$, 使$f_i^m \to f_i$, 易见$f_i(x) = 1$. 再依定义1的条件Ⅰ, $f_1, f_2, \cdots, f_{k+1}$线性相关, 不妨设$f_{k+1} = \sum_{i=1}^{k} a_i f_i$, 应用$k$-维体积的连续性, 有

$$\lim_{m\to\infty} A(f_1^m, f_2^m, \cdots, f_{k+1}^m) = A(f_1, f_2, \cdots, f_{k+1})$$

$$= \sup\left\{\begin{vmatrix} f_1(x) & f_2(x) & \cdots & (\sum_{i=1}^{k} a_i f_i)(x) \\ F_1(f_1) & F_1(f_2) & \cdots & F_1(\sum_{i=1}^{k} a_i f_i) \\ \cdots & \cdots & \cdots & \cdots \\ F_k(f_1) & F_k(f_2) & \cdots & F_k(\sum_{i=1}^{k} a_i f_i) \end{vmatrix}; F_i \in U(X^{**}), i = 1, 2, \cdots, k\right\}$$

$$= \sup\left\{\sum_{i=1}^{k} a_i \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_i(x) \\ F_1(f_1) & F_1(f_2) & \cdots & F_1(f_i) \\ \cdots & \cdots & \cdots & \cdots \\ F_k(f_1) & F_k(f_2) & \cdots & F_k(f_i) \end{vmatrix}; \begin{matrix} F_i \in U(X^{**}), \\ i = 1, 2, \cdots, k \end{matrix}\right\} = 0,$$

这与(a)式矛盾.

设(1)成立, 类似于文[2]定理5的证明, 可以导出x是X的k-强光滑点.

引理3[3] X局部k-一致凸的充要条件是对任何$x \in S(X)$, $\{x_i^n\} \subset X$, $i = 1, 2, \cdots, k$, 当$\|x_i^n\| \to 1$, $\|x + x_1^n + \cdots + x_k^n\| \to k + 1$时, $A(x, x_1^n, \cdots, x_k^n) \to 0$.

引理4 (局部自反原理)[4] 设X_0^*, X_0^{**}分别是X^*和X^{**}的有限维子空间, 则对任何$\epsilon \in (0, 1)$, 存在一一的有界线性算子$T: X_0^{**} \to X$, 使

(1) $(1 - \epsilon)\|F\| \leq \|T(F)\| \leq (1 + \epsilon)\|F\|$; $F \in X_0^{**}$;

(2) $f(T(F)) = F(f)$; $f \in X_0^*$, $F \in X_0^{**}$;

(3) $T(x) = x$; $x \in X \cap X_0^{**}$.

下面给出本文的主要结果.

定理5 若X局部k-一致凸, 则对每个$x \in S(X)$, $f \in \Sigma(x)$是X^*的k-强光滑点.

证明 如果有$x \in S(X)$, $f \in \Sigma(x)$不是X^*的k-强光滑点, 由引理2知, 存在$\{F_i^n\} \subset S(X^{**})$, $F_i^n(f) \to 1$, $i = 1, 2, \cdots, k+1$, 及$\epsilon_0 > 0$和$\{n\}$的子列$\{m\}$, 使$A(F_1^m, F_2^m, \cdots, F_{k+1}^m) \geq \epsilon_0$. 首先, 对每个$m$, 依$k$-维体积的定义, 有$G_1^m, G_2^m, \cdots, G_k^m \in U(X^{***})$, 满足

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ G_1^m(F_1^m) & G_1^m(F_2^m) & \cdots & G_1^m(F_{k+1}^m) \\ \cdots & \cdots & \cdots & \cdots \\ G_k^m(F_1^m) & G_k^m(F_2^m) & \cdots & G_k^m(F_{k+1}^m) \end{vmatrix} \geq \epsilon_0/2,$$

再利用Goldstine-Weston稠密性定理[3], 容易知道, 存在$f_1^m, f_2^m, \cdots, f_k^m \in U(X^*)$, 使

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$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ F_1^m(f_1^m) & F_2^m(f_1^m) & \cdots & F_{k+1}^m(f_1^m) \\ \cdots & \cdots & \cdots & \cdots \\ F_1^m(f_k^m) & F_2^m(f_k^m) & \cdots & F_{k+1}^m(f_k^m) \end{vmatrix} \geq \epsilon_0/4 \tag{b}$$

其次, 对每个m, 令$X_m^{**} = \mathrm{span}\{x, F_1^m, \cdots, F_{k+1}^m\}$, $X_m^* = \mathrm{span}\{f_1^m, f_2^m, \cdots, f_k^m\}$, 据局部自反原理, 存在一一的有界线性算子$T_m: X_m^{**} \to X$, 满足

(1) 对每个$i(1 \leq i \leq k+1)$, $1 - 1/m \leq \|T_m(F_i^m)\| \leq 1 + 1/m$, $(1 - 1/m)\|x + F_1^m + \cdots + F_{i-1}^m + F_{i+1}^m + \cdots + F_{k+1}^m\| \leq \|T_m(x + F_1^m + \cdots + F_{i-1}^m + F_{i+1}^m + \cdots + F_{k+1}^m)\| \leq (1 + 1/m)\|x + F_1^m + \cdots + F_{i-1}^m + F_{i+1}^m + \cdots + F_{k+1}^m\|$;

(2) $f_j^m(T_m(F_i^m)) = F_i^m(f_j^m)$, $i = 1, 2, \cdots, k+1$, $j = 1, 2, \cdots, k$;

(3) 对任何m, $T_m(x) = x$.

记$x_i^m = T_m(F_i^m)$, 则$\|x_i^m\| \to 1$, 注意到$\|x + F_1^m + \cdots + F_{i-1}^m + F_{i+1}^m + \cdots + F_{k+1}^m\| \to k+1$, 有$\|x + x_1^m + \cdots + x_{i-1}^m + x_{i+1}^m + \cdots + x_{k+1}^m\| \to k+1$. 因$X$局部$k$-一致凸, 由引理3

$$A(x, x_1^m, \cdots, x_{i-1}^m, x_{i+1}^m, \cdots, x_{k+1}^m) \to 0, \quad i = 1, 2, \cdots, k+1. \tag{c}$$

最后, 对任何m及$f_1, f_2, \cdots, f_k \in U(X^*)$, 记$A(x_1^m, x_2^m, \cdots, x_{k+1}^m)$中的行列式为

$$D = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ f_1(x_1^m) & f_1(x_2^m) & \cdots & f_1(x_{k+1}^m) \\ \cdots & \cdots & \cdots & \cdots \\ f_k(x_1^m) & f_k(x_2^m) & \cdots & f_k(x_{k+1}^m) \end{vmatrix},$$

不妨设$D \neq 0$(否则是平凡的), 于是关于$t_1, t_2, \cdots, t_{k+1}$的线性方程组

$$\begin{cases} t_1 + t_2 + \cdots + t_{k+1} = 1, \\ f_1(x_1^m)t_1 + f_1(x_2^m)t_2 + \cdots + f_1(x_{k+1}^m)t_{k+1} = f_1(x), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_k(x_1^m)t_1 + f_k(x_2^m)t_2 + \cdots + f_k(x_{k+1}^m)t_{k+1} = f_k(x). \end{cases}$$

有唯一的解: $t_1 = D_1/D$, $t_2 = D_2/D$, $\cdots$, $t_{k+1} = D_{k+1}/D$, 其中

$$D_i = \begin{vmatrix} 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ f_1(x_1^m) & \cdots & f_1(x_{i-1}^m) & f_1(x) & f_1(x_{i+1}^m) & \cdots & f_1(x_{k+1}^m) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ f_k(x_1^m) & \cdots & f_k(x_{i-1}^m) & f_k(x) & f_k(x_{i+1}^m) & \cdots & f_k(x_{k+1}^m) \end{vmatrix}, \quad i = 1, 2, \cdots, k+1.$$

将其解代入第一个方程得: $D = \sum_{i=1}^{k+1} D_i$. 从而, 据(c)式有,

$$\begin{aligned} A(x_1^m, x_2^m, \cdots, x_{k+1}^m) &= \sup\{\sum_{i=1}^{k+1} D_i; f_j \in U(X^*), j = 1, 2, \cdots, k\} \\ &\leq \sum_{i=1}^{k+1} \sup\{D_i; f_j \in U(X^*), j = 1, 2, \cdots, k\} \\ &= \sum_{i=1}^{k+1} A(x, x_1^m, \cdots, x_{i-1}^m, x_{i+1}^m, \cdots, x_{k+1}^m) \to 0. \end{aligned}$$

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但另一方面, 依(2) 及(b) 式$A(x_1^m, x_2^m, \cdots, x_{k+1}^m) \geq \epsilon_0/4$, 矛盾.

应用定理 5, 容易得到

推论 6 X 局部一致凸的充要条件为对每个$x \in S(X)$, $f \in \Sigma(x)$ 是X^* 的强光滑点且X 具有(WM) 性质(即对任何$x \in S(X)$, $f \in \Sigma(x)$, $\{x_n\} \subset S(X)$, 当 $\|x + x_n\| \to 2$ 时, 存在$\{x_n\}$ 的子列$\{x_m\}$, 使$f(x_m) \to 1$. 见[5]).

文[6] 导出了局部k-一致凸空间具有(H) 性质(即对任何$x \in S(X)$, $\{x_n\} \subset S(X)$, 当$x_n \xrightarrow{w} x$ 时, $x_n \to x$), 事实上, 有如下更强的结论

定理 7 若X 局部k-一致凸, 则每个$x \in S(X)$ 是$U(X^{**})$ 的弱*-范数连续性点(即对任何网$\{F_\alpha\} \subset U(X^{**})$, 当$F_\alpha \xrightarrow{w^*} x$ 时, $F_\alpha \to x$. 参见[7]).

证明 由定理 5 知, 对每个$x \in S(X)$, $f \in \Sigma(x)$ 是X^* 的k-强光滑点, 而k-强光滑蕴含(S) 性质(即定义 1 中的条件 II. 见[5]), 且x 作为$S(X^{**})$ 中的点在f 处达到范数, 据文[7] 定理 1, x 是$U(X^{**})$ 的弱*-范数连续性点

推论 8 若X 局部k-一致凸, 则每个$x \in S(X)$ 是$U(X^{**})$ 的PC 点(即对任何网$\{F_\alpha\} \subset U(X^{**})$, 当$F_\alpha \xrightarrow{w} x$ 时, $F_\alpha \to x$. 参见[7]).

推论 9 若X 局部k-一致凸, 则每个$x \in S(X)$ 是$U(X)$ 的PC 点

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The Dual Space of the Local k-Uniform Rotund Space

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Abstract

In this paper, we prove that if Banach space X is local k-uniform rotund, then for every $x \in S(X)$, $f \in \Sigma(x)$ is a k-strong smooth point of X^*. We also obtain some properties of the local k-uniform rotund space.

Keywords local k-uniform rotund, k-strong smooth, k-dimensional volume.

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