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On Zero-One Symmetric and Normal Matrices*

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Abstract We present additional equivalent conditions on the existence of a 0-1 symmetric matrix with given row sum vector. The case of a 0-1 normal matrix is also considered.

Keywords incidence matrix, majorization, normal matrix, row sum vector.

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1. Introduction

A 0-1 matrix is a matrix all of whose entries are 0's and 1's. Let A be a 0-1 matrix of size $m \times n$, let r_i and s_j be the sum of row i and column j of A, respectively, $i = 1, \cdots, m$, $j = 1, \cdots, n$. Denote by $R(A) = (r_1, \cdots, r_m)$ and $S(A) = (s_1, \cdots, s_n)$ the *row* and *column sum vectors* of A. It is immediate that $R(A)$ and $S(A)$ are integral vectors and $r_1 + \cdots + r_m = s_1 + \cdots + s_n$.

Before preceeding we say that vector $x = (x_1, \cdots, x_n)$ is *majorized* by vector $y = (y_1, \cdots, x_n)$, written $x \prec y$, if $\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}$, $k = 1, \cdots n-1$, $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, when the components of x and y are rearranged in monotone order $x_{[1]} \ge \cdots \ge x_{[n]}$, $y_{[1]} \ge \cdots \ge y_{[n]}$.

For $R = (r_1, \cdots, r_m)$ and positive integer n such that $r_i \le n$, $i = 1, \cdots, m$, we define the conjugate R^* of R by $R^* = (r_1^*, \cdots, r_n^*)$, $r_i^* = |\{k: r_k \ge i, k = 1, \cdots, m\}|$, $i = 1, \cdots, n$. It is easy to see [2, p. 175] that for every positive integer $k \le n$

$$\sum_{i=1}^{k} r_i^* = \sum_{i=1}^{m} \min\{r_i, k\}.$$

Given integral vectors $R = (r_1, \cdots, r_m)$ and $S = (s_1, \cdots, s_n)$, it has long been an interesting problem to determine the existence of an $m \times n$ 0-1 matrix which has the prescribed row and column sum vectors R and S. The question was first discussed by Gale[3] and

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Ryser[9], and later by many other authors (see, e. g., [1] and the references therein, and [11]).

A remarkable result known as the Gale-Ryser Theorem [2, p. 176] characterizes the relationship of R and S.

Theorem 1 (Gale-Ryser) *There exists an $m \times n$ 0-1 matrix with row and column vectors equal to R and S, respectively, if and only if $S \prec R^*$.*

With R and S satisfying $S \prec R^*$, a 0-1 matrix of respective row and column vectors R and S and of particular structure is often desired. The existence of 0-1 symmetric matrices (in a more general setting) has been studied. The following result can be found in [2, p. 181].

Theorem 2 *Let $R = (r_1, \cdots, r_n)$ be a monotone vector each of whose components does not exceed n. Then the following conditions are equivalent.*

(*a*) *There exists an n-square 0-1 symmetric matrix with row sum vector R;*

(*b*) $\sum_{i=1}^{k} r_i \le kt + \sum_{j=t+1}^{n} r_j, \quad k, t = 1, \cdots, n;$

(*c*) $R \prec R^*$.

The aim of this note is to give more equivalent conditions on the existence of 0-1 symmetric matrices, and to discuss the case of 0-1 normal matrices. In particular, when $R = S$ a weaker condition can substitute that of the Gale-Ryser Theorem (see Theorem 3 (7)). It is noted that the criteria for integer sequences being graphic are discussed in [10].

2 0-1 Symmetric Matrices

In what follows, let $R = (r_1, \cdots, r_n)$ be a nonzero integral vector each of whose components is no more than n. Since the existence of the matrices to be considered does not change under permutation, we may assume that $r_1 \ge \cdots \ge r_n$. let $R^* = (r_1^*, \cdots, r_n^*)$ be the conjugate of R, and let k, t be positive integers no more than n.

Theorem 3 *Each of the following conditions is equivalent to (a) in Theorem 2.*

(1) $\sum_{i=1}^{k} r_i \le k^2 + \sum_{j=k+1}^{n} \min\{k, r_j\};$

(2) $\sum_{i=1}^{k} r_i \le kt + \sum_{j=t+1}^{n} \min\{k, r_j\}, \quad k \le t;$

(3) $\sum_{i=1}^{k} r_i \le kt + \sum_{j=t+1}^{n} r_j, \quad k \le t;$

(4) $\sum_{i=1}^{k} r_i \le \sum_{i=1}^{k} \min\{t, r_i\} + \sum_{j=t+1}^{n} r_j, \quad k \le t;$

(5) $\sum_{i=1}^{k} r_i \le \sum_{i=1}^{k} \min\{k, r_i\} + \sum_{j=k+1}^{n} r_j;$

(6) $\sum_{i=1}^{k} \max\{k, r_i\} \le k^2 + \sum_{j=k+1}^{n} r_j;$

(7) $\sum_{i=1}^{k} r_i \le \sum_{i=1}^{k} r_i^*, \quad 1 \le k \le f$, where $f = \max\{i: r_i \ge i\}$.

(8) $\sum_{i=1}^{k} r_i \le kt + \sum_{j=t+1}^{n} \min\{k, r_j\};$

(9) $\sum_{i=1}^{k} r_i \le \sum_{i=1}^{k} \min\{t, r_i\} + \sum_{j=t+1}^{n} r_j;$

(10) $\sum_{i=1}^{k} \max\{t, r_i\} \le kt + \sum_{j=t+1}^{n} r_j;$

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(11) $\sum_{i=1}^{k}\max\{t, r_i\} \le kt + \sum_{j=t+1}^{n} r_j, \quad k \le t;$

proof We show that $(a) \Rightarrow (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7) \Rightarrow (c)$. Since (a) is equivalent to (c), all of the above will be equivalent. Similarly one can prove that the rest conditions are equivalent to (b).

(a) $\Rightarrow$ (1): Considering the first k rows of the 0-1 symmetric matrix, we have

$$\sum_{i=1}^{k} r_i \le k^2 + \sum_{j=k+1}^{n}\min\{k, s_j\} = k^2 + \sum_{j=k+1}^{n}\min\{k, r_j\}.$$

(1) $\Rightarrow$ (2): If $k = t$, there is nothing to prove. For $k < t$,

$$\sum_{i=1}^{k} r_i \le k^2 + \sum_{j=k+1}^{n}\min\{k, r_j\} = k^2 + \sum_{j=k+1}^{t}\min\{k, r_j\} + \sum_{j=t+1}^{n}\min\{k, r_j\}$$

$$\le k^2 + k(t-k) + \sum_{j=t+1}^{n}\min\{k, r_j\} = kt + \sum_{j=t+1}^{n}\min\{k, r_j\}.$$

(2) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (4): If $r_i \ge t$, $i = 1, \cdots, k$, then (3) and (4) are the same. Otherwise we may assume that $r_m \ge t$ and $r_{m+1} < t$ for some positive integer $m < k$. Then

$$\sum_{i=1}^{k} r_i = \sum_{i=1}^{m} r_i + \sum_{i=m+1}^{k} r_i \le mt + \sum_{j=t+1}^{n} r_j + \sum_{i=m+1}^{k} r_i \quad \text{(use (3))}$$

$$= \sum_{j=t+1}^{n} r_j + \sum_{i=m+1}^{k} r_i + \sum_{i=1}^{m} t = \sum_{j=t+1}^{n} r_j + \sum_{i=m+1}^{k}\min\{t, r_i\} + \sum_{i=1}^{m}\min\{t, r_i\}$$

$$= \sum_{j=t+1}^{n} r_j + \sum_{i=1}^{k}\min\{t, r_i\}.$$

(4) $\Rightarrow$ (5): Set $k = t$.

(5) $\Rightarrow$ (6): In fact each "$\Rightarrow$" below can be replaced with "$\Leftrightarrow$":

$$\sum_{i=1}^{k} r_i \le \sum_{i=1}^{k}\min\{k, r_i\} + \sum_{j=k+1}^{n} r_j \Rightarrow \sum_{i=1}^{k} r_i + \sum_{i=1}^{k}\max\{-k, -r_i\} \le \sum_{j=k+1}^{n} r_j$$

$$\Rightarrow \sum_{i=1}^{k}\max\{0, r_i - k\} \le \sum_{j=k+1}^{n} r_j \Rightarrow \sum_{i=1}^{k}(\max\{k, r_i\} - k) \le \sum_{j=k+1}^{n} r_j$$

$$\Rightarrow \sum_{i=1}^{k}\max\{k, r_i\} \le k^2 + \sum_{j=k+1}^{n} r_j.$$

(6) $\Rightarrow$ (7): Suppose that $\sum_{i=1}^{m} r_i > \sum_{i=1}^{m} r_i^*$ for some $m \le f$. Then $r_m \ge r_f \ge f \ge m$. Let t be the positive integer such that $r_{t+1} < m \le r_t$. Then $t \ge m$, and $r_i \ge m$ when $i \le t$ and $r_i < m$

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when $i > t$. Therefore

$$\sum_{i=1}^{m} r_i^* = \sum_{i=1}^{n} \min\{r_i, m\} = mt + \sum_{j=t+1}^{n} r_j$$

$$\geq mt + \sum_{i=1}^{t} \max\{t, r_i\} - t^2 \quad (by\ (6))$$

$$= \sum_{i=1}^{m} \max\{t, r_i\} + \sum_{i=m+1}^{t} \max\{t, r_i\} - t(t-m)$$

$$\geq \sum_{i=1}^{m} r_i + t(t-m) - t(t-m) = \sum_{i=1}^{m} r_i.$$

This is a contradiction.

(7) ⇒ (c): Suppose that m is the smallest positive integer satisfying $m > f$ and

$$\sum_{i=1}^{m} r_i > \sum_{i=1}^{m} r_i^*.$$

Then $r_m < m$. Let t be such that $r_{t+1} < m \leq r_t$. Then $t < m$, and $r_i \geq m$ when $i \leq t$ and $r_i < m$ when $i > t$. Thus $\sum_{i=1}^{m} r_i^* = \sum_{i=1}^{n} \min\{r_i, m\} = mt + \sum_{i=t+1}^{n} r_i < \sum_{i=1}^{m} r_i$ (assumption) or $mt + \sum_{i=t+1}^{m} r_i + \sum_{i=m+1}^{n} r_i < \sum_{i=1}^{t} r_i + \sum_{i=t+1}^{m} r_i$. Hence $mt + \sum_{i=m+1}^{n} r_i < \sum_{i=1}^{t} r_i$.

On the other hand,

$$\sum_{i=1}^{t} r_i^* = \sum_{i=1}^{n} \min\{r_i, t\} \leq mt + \sum_{i=m+1}^{n} r_i.$$

It follows that $\sum_{i=1}^{t} r_i > \sum_{i=1}^{t} r_i^*$, a contradiction to m being the smallest.

Note that (3) and (7) are apparently weaker than (b) and (c), respectively. Thus when $R = S$, the condition of the Gale-Ryser Theorem can be replaced by the weaker condition (7). For example, if $R = S = (5, 3, 2, 1, 1, 1)$, then $f = 2$ and one may just check two inequalities in (7) instead of six.

3. 0-1 Normal Matrices

A real square matrix A is said to be normal if A commutes with its transpose, denoted A^T; that is $AA^T = A^TA$. It is immediate that the row and column sum vectors of any 0-1 normal matrix A are identical, i.e., $R(A) = S(A)$.

Normal matrices are frequently seen in many subjects of mathematics. In design theory or combinatorial matrix theory, one considers the incidence matrices of (v, k, λ)-designs, the 0-1 matrices each of whose rows and columns has k 1's and any two distinct rows (columns) have exactly λ 1's in common columns (rows). Equivalently $AA^T = (k-\lambda)I + \lambda J$, where I is the identity matrix and J is the matrix all of whose entries are 1's. It can be shown [2, p. 14, 5, p. 368, 8, p. 72, or 9, p. 103] that such a matrix A is normal. Another

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kind of important normal matrix is a circulant matrix [8, p. 66]. Any 0-1 circulant matrix A is a normal matrix whose row (column) sum vector consists of the same component; namely, $R(A) = S(A) = (k, k, \cdots, k)$, where k is the number of 1's in the first row. In addition, a 0-1 tournament matrix A [2, p. 54], a matrix satisfying $A + A^T = J - I$ is normal when $R(A) = S(A) = (t, t, \cdots, t)$ for some t. For more on general normal matrices, see [4, 6, 7, and 12].

The result below follows immediately from the Theorems 1 and 2. The proof in [2, p. 180] (in a more general setting) is a graphic one. We show (f) ⇒ (e) in a different approach, in which one will see how to construct a 0-1 symmetric matrix from a matrix with equal row and column sum vectors. The technique in the proof is often used in combinatorial matrix theory [9, p. 67] when row or column sum vectors of 0-1 matrices are investigated. In addition, the following theorem reveals that it is impossible to distinguish 0-1 symmetric matirces from normal ones with row and column sum vectors.

Theorem 4 *The following are equivalent:*

(*d*) *There exists a* 0-1 *normal matrix* A *such that* $R(A) = R$;

(*e*) *There exists a* 0-1 *symmetric matrix* B *such that* $R(B) = R$;

(*f*) *There exists a* 0-1 *matrix* C *such that* $R(C) = S(C) = R$.

proof We show only that (f) implies (e) by induction.

Suppose that $C = (c_{ij})$ is not symmetric. We may assume that for some p and q

$$c_{1p} = 1, \quad c_{1q} = 0, \quad c_{p1} = 0, \quad c_{q1} = 1.$$

First, there must be a t such that $c_{pt} = 1$ and $c_{qt} = 0$, or $c_{tp} = 0$ and $c_{tq} = 1$. Otherwise $c_{pt} \leq c_{qt}$, $t = 2, \cdots, n$, which leads to $r_p < r_q$, and $c_{sp} \geq c_{sq}$, $s = 2, \cdots, n$, which leads to $s_p > s_q$. Therefore $r_p < r_q = s_q < s_p = r_p$, a contradiction.

We now interchange the 0's and 1's in the intersections of rows p, q and columns 1, t, or columns p, q and rows 1, t. Notice that such an interchange reduces the number of different entries in the first row and first column of C without affecting the row and column sum vectors of C. By induction on the number of distinct entries in the first row and first column of C, we can have a matrix whose first column is the transpose of the first row. By induction on the size of matrices, we see that a symmetric matrix can be obtained from C. □

The investigation of 0-1 normal matrices, when considered as incidence matrices, may be interpreted as follows.

Let $X = \{x_1, x_2, \cdots, x_n\}$ be a set of n elements, let $X_1, X_2, \cdots, X_n$ be n subsets (not necessarily distinct) of X, and let $A = (a_{ij})$ be the incidence matrix defined by

$$a_{ij} = \begin{cases} 1 & \text{if } x_j \in X_i \\ 0 & \text{otherwise,} \end{cases} \qquad i, j = 1, \cdots, n.$$

Let $AA^T = B$ and $A^TA = C$. Then the (i, j)-entry of B is the same as $|X_i \cap X_j|$, the cardinality of $X_i \cap X_j$, and the (i, j)-entry of C is equal to $|x_i \wedge x_j|$, the number of subsets $X_1, X_2, \cdots, X_n$ which contain x_i and x_j simultaneously.

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It is readily seen that A is normal if and only if $|X_i \cap X_j| = |x_i \wedge x_j|$ for each pair of i and j.

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关于(0, 1)的对称和规范矩阵

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摘　要

给出更多的对于给定行和向量存在(0, 1)对称矩阵的等价条件，同时也讨论了(0, 1)规范矩阵的情形

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