Journal of Mathematical Research & Exposition
Vol. 18, No. 2, 173- 177, May 1998

π-Frattini Subgroup and π-Local Formation*

Chen Zhongmu†
(Inst. of Math., Southwest China Normal University, Chongqing 400715)

Abstract In this paper, our purpose is to make the results about π-Frattini subgroup more accurate, and to extend Gaschüz Theorem about nilpotency to π-locally defined formation. We come to

Theorem *Let G be a finite group, H a subnormal subgroup of G. If $H/H \cap \Phi(G)O_{\pi'}(G) \in \mathbf{F}$, then $H \in \mathbf{F}_\pi$, where $\mathbf{F}_\pi$ is π-solvable π-locally defined formation.*

Keywords π-Frattini subgroup, π-locally defined formation, Gaschüz theorem.

Classification AMS(1991) 20D20, 20F17/CCL O152.1

All groups mentioned in this paper are finite groups. For a group G, $H \lhd\lhd G$ denotes H is a subnormal subgroup of G; $M <\cdot G$ denotes M is a maximal subgroup of G. All other notations are standard.

It is well-known that the Frattini subgroup of a group plays a very important role in researching locally defined formation. Similar to Frattini subgroup, several kinds of subgroups are defined and learned (refer to [1]). Some of them are related to formation, such as $\Phi_p(G)$ [2, Appedix C, §4], $\varphi_p(G)$ defined by Deskin (1, p418, Def, 3.3.1]), and $\Phi_\pi(G)$ with more general meaning[3]. The following definition is slightly different to the definition of $\Phi_\pi(G)$ in [3].

Suppose π is a set of primes, G is a group. If G has a maximal subgroup M, such that $|G:M|$ is a π-number. Define

$$\Phi_\pi(G) = \cap \{M \mid M <\cdot G, |G:M| \text{ is a } \pi\text{-number} \tag{1}$$

Otherwise, define $\Phi_\pi(G) = \Psi_\pi(G) = \Phi(G)O_{\pi'}(G)$.

Remark In [3], $\Phi_\pi(G) = \{M \mid M <\cdot G > (|G:M|, p) = 1, p \in \pi\}$. This $\Phi_\pi(G)$ may be viewed as $\Phi_\pi(G)$ in (1) if we take $\pi = \pi'$ and $\Phi_\pi(G) = G$ while the maximal subgroup as in (1) does not exist.

If $\pi = p'$, then $\Phi_\pi(G)$ in (1) is $\varphi_p(G)$ defined by Deskin[1]. If $\pi = \{p\}$, then $\Phi_\pi(G) = \Phi_p(G)$[2].

* Received October 7, 1996. Project supported by the National Natural Science Foundation of China.
† Saldy, the passed away on February 16, 1998.

In this paper, we shall extend the results about $\Phi_\pi(G)$ obtained before and make them more accurate. And we shall unify the Gaschütz Theorem about nipotency and theorems about p-closed groups in [1], and extend them to π-local formation.

Lemma 1 1) *$\Phi_\pi(G)$ char G*; 2) $\Phi(G)\le\Phi_\pi(G)$; 3) $\Phi_\pi(G/N)\ge\Phi_\pi(G)N/N$, *equality appears if* $N\le\Phi_\pi(G)$.

Proof The Lemma follows by the definition of $\Phi_\pi(G)$.

Lemma 2 1) $O_{\pi'}(G)\le\Phi_\pi(G)$; 2) $O_\pi(\Phi_\pi(G))\le\Phi(G)$. *Further, if $\Phi_\pi(G)$ is a π-group, then* $\Phi_\pi(G)\le\Phi(G)$, *so* $\Phi_\pi(G)=\Phi(G)$.

Proof 1) If $O_{\pi'}(G)\not\le\Phi_\pi(G)$, then there exists $M<\cdot G$, such that $|G:M|$ is a π'-number and $M\not\ge O_{\pi'}(G)$. So $G=MO_{\pi'}(G)$. Therefore

$$|G:M|=|MO_{\pi'}(G):M|=|O_{\pi'}(G):M\cap O_{\pi'}(G)|$$

is a π-number, a contradiction to choose of M.

2) If $O_\pi(\Phi_\pi(G))\not\le\Phi(G)$, then there exists $M<\cdot G$, such that $M\not\ge O_\pi(\Phi_\pi(G))$, $G=MO_\pi(\Phi_\pi(G))$. Therefore $|G:M|=|O_\pi(\Phi_\pi(G)):O_\pi(\Phi_\pi(G))\cap M|$ is a π-number. So $M\ge\Phi_\pi(G)\ge O_\pi(\Phi_\pi(G))$, a contradiction to choose of M.

Theorem 1 1) $\Psi_\pi(G)=\Phi(G)O_{\pi'}\le\Phi_\pi(G)$.

2) If $\Phi_\pi(G)$ satisfies condition $C_{\pi'}$, that is, the π'-Hall subgroups of $\Phi_\pi(G)$ exists and are conjugate in $\Phi_\pi(G)$. Then

$$\Phi_\pi(G)=\Psi_\pi(G)=\Phi(G)_\pi\times O_{\pi'}(G).$$

Proof 1) Follows from Lemma 1, 2) and Lemma 2. 1).

2) Suppose $\Phi_\pi(G)$ satisfies $C_{\pi'}$. Let K be a π'-Hall subgroup of $\Phi_\pi(G)$. Then $G=N_G(K)\Phi_\pi(G)$ by Frattini argument. If $N_G(K)<G$, then there exists $M<\cdot G$, such that $N_G(K)\le M$, $G=M\Phi_\pi(G)$,

$$|G:M|=|\Phi_\pi(G):M\cap\Phi_\pi(G)|.$$

Since M contains K and $M\cap\Phi_\pi(G)$, M contains a π'-Hall subgroup of $\Phi_\pi(G)$, $|\Phi_\pi(G):M\cap\Phi_\pi(G)|$ is a π-number. So $M\ge\Phi_\pi(G)$, $G=M$, a contradiction. Hence $K\le G$, $K\le O_{\pi'}(G)$. Again from Lemma 2, 1), we have $K=O_{\pi'}(G)$. By Lemma 1, 3), $\Phi_\pi(G/O_{\pi'}(G))=\Phi_\pi(G)/O_{\pi'}(G)$ is a π-group, by Lemma 2, 2),

$$\Phi_\pi(G/O_{\pi'}(G))=\Phi(G/O_{\pi'}(G))_\pi\ge(\frac{\Phi(G)O_{\pi'}(G)}{O_{\pi'}(G)}=\frac{\Phi(G)O_{\pi'}(G)}{O_{\pi'}(G)},$$

which implies $\Phi_\pi(G)\ge\Phi(G)_\pi O_{\pi'}(G)$. By Lemma 1, 2) and Lemma 2, 1), $\Phi_\pi(G)=\Phi(G)O_{\pi'}(G)$. The Theorem 1 follows

Corollary 1 1) *If $\Phi_\pi(G)$ satisfies $C_{\pi'}$, then* $\Phi_\pi(G)/O_{\pi'}(G)=\Phi(G/O_{\pi'}(G))$.

2) $\Phi_{p'}(G)=\Phi(G)O_p(G)=\Phi(G)_{p'}\times O_p(G)$ *and is nilpotent.*

Proof Since the $(p)'$-Hall subgroup of $\Phi_{p'}(G)$ is a p-Sylow subgroup, $\Phi_{p'}(G)$ satisfies C_p.

Theorem 1 and its corallary are a kind of extension and accuratization of Theorem 3.3.7 in [1] and Theorem 4.1, 4.2 in [2, Appedix c, §4]. By [3], $\Phi_{p'}(G)$ and $\Phi_{\{p,q\}'}(G)$ are solvable, and $\Phi_{p'}(G)$ is an extension of a p-group by a nilpotent group.

$\Phi_2(S_5)=A_5$ implies that condition $C_{\pi'}$ cannot be dropped. But it is worth to discuss if $C_{\pi'}$ can be weaken as $E_{\pi'}$, that is the π-Hall subgroup of $\Phi_\pi(G)$ exists

$\Phi_2(S_5)=\Phi_{\{3,5\}'}(S_5)$ implies $\Phi_{\{p,q\}'}(G)$ may not be solvable. So result about $\Phi_{\{p,q\}'}(G)$ in [3] is incorrect.

Frattini subgroup $\Phi(G)$ in the theory of locally defined formation has a most important property.

Suppose $\mathbf{F}$ is a locally defined formation, then "$G/\Phi(G)\in\mathbf{F}$ if and only if $G\in\mathbf{F}$."

If $\mathbf{F}=\mathbf{N}$ (formation of nilpotent groups), Gaschütz extended above property:

Let D,M be normal subgroups of $G, D\lhd M$, $D\leq\Phi(G)$. If $M/D\in\mathbf{N}$, then $M\in\mathbf{N}$, ([4, III, Theorem 5]).

Janko extended Gaschütz Theorem as

Let $H\lhd\lhd G$. If $H/H\cap\Phi(G)\in\mathbf{N}$, then $H\in\mathbf{N}$ ([1, p423, Cor. 3.3.17]).

This result is also be extended in [1] as:

Let $\mathbf{F}$ be p-closed group formation. If $H\lhd\lhd G$, $H/H\cap\Phi_{p'}(G)\in\mathbf{F}$, then $H\in\mathbf{F}$ ([1, p422, Th. 3.3.16]).

Here we shall unify above results and extend them to local formation.

Let $\mathbf{F}_\pi$ be π-solvable π-locally defined formation. For any $f(p)\in\{f(p)\}$ be defined formation, $f(p)\neq\emptyset$, and so $\mathbf{F}_\pi\supset\mathbf{N}$.

Theorem 2 *Suppose* $H\lhd\lhd G$, $H/H\cap\Psi_\pi(G)\in\mathbf{F}_\pi$, *then* $H\in\mathbf{F}_\pi$

We need the following Lemmas for proving this Theorem.

Lemma 3 *Suppose* $H\lhd G, K\lhd H$ *such that* K *satisfies property* σ, σ *is closed for product of normal subgroups. Then* G *contains a normal* σ-*subgroup* $L, L\geq K$.

Proof Since $K^x\lhd H$, $\forall x\in G$, $L=\cup_{x\in G}K^x$ as desired.

Surely, in this Lemma condition "$H\lhd G$" can be changed into "$H\lhd\lhd G$."

π-group, π-solvable group, p-closed group, p-nilpotent group and so on are closed for product of normal subgroups

Lemma 4 *Let* $G=MN$, $M\cap N=1$, N *an abelian normal* p-*subgroup of* G, *if* G *has a normal* p-*solvable subgroup* L, *such that* $L\geq N$, $L\neq N$. *Then any compliment of* N *is conjugate to* M.

Proof Assume N_1 is a minimal normal p-subgroup of G, such that $N\geq N_1$, $N_1\neq N$. Set

$$\overline{G}=G/N_1,\ \overline{M}=MN_1/N_1,\ \overline{N}=N/N_1.$$

Hence $\overline{G}=\overline{MN}$, $\overline{N}<\overline{L}\lhd\overline{G}$. Using induction on $\overline{G}$, we have the compliment of $\overline{N}$ is conjugate to $\overline{M}$.

If $N>N_1$, then the compliment of N is conjugate to M by induction on MN_1. Hence we may assume N is the minimal normal subgroup of G. It is easy to show $M<\cdot G$. Put $M_G=$

$\cap_{x\in G} M^x$. Again set $\overline{G} = G/M_G$, $\overline{M} = M/M_0$, $\overline{L} = LM_G/M_G$, $\overline{N} = NM_G/M_G$. Hence $\overline{M} <\cdot \overline{G}$, $\overline{M}_{\overline{G}} = 1$, $O(\overline{G}) = \overline{N}$. If $O_p(\overline{G}) > \overline{N}$, then $D = O_p(\overline{G}) \cap \overline{M} \lhd \overline{M}$. But $\overline{N}$ is a p-group and $N_{\overline{N}}(D) > D$. Therefore $N_{\overline{N}}(D) > \overline{M}$, $N_{\overline{G}}(D) = \overline{G}$, a contradiction to $M_{\overline{G}} = 1$. Since $\overline{N} < \overline{L} \lhd \overline{G}$, $1 \neq \overline{H} = \overline{L} \cap \overline{M}$, which is a normal p-solvable subgroup of $\overline{M}$. We have proved $O_p(\overline{M}) = 1$ early, which implies $O_p(\overline{H}) = 1$, $O_p'(\overline{H}) \neq 1$. Hence $O_p'(\overline{M}) \neq 1$. By [5, p. 276, Th. 11, 14] the compliments of $\overline{N}$ in G are conjugate which leads to the compliments of N in G are conjugate. This is the end of the proof.

Remark The condition of the Theorem in [5] is $O_q(M) \neq 1$. But it can be changed into $O_p'(M) \neq 1$ by using Schur-Zassenhaus Theorem instead of Sylow Theorems, the Theorem still holds by the same proof.

Lemma 5 *Let $G = HA$, $A \lhd G$, A an abelian group. If $HA \in \mathbf{F}_\pi$, then $H \in \mathbf{F}_\pi$*

Proof (Induction on $|G|$) Assume $H \leq G$, $H \neq G$, $H \leq M <\cdot G$. Set $D = M \cap A$. Then $D \lhd \langle M, A\rangle = G$.

If $D = 1$, then $H \cap A \leq M \cap A = 1$. Hence $H \cap A = 1$, which implies $H = H/H \cap A \cong HA/A \in \mathbf{F}_\pi$

If $D \neq 1$, let N be a minimal normal subgroup of G, such that $N \leq D$. Using induction on $\overline{G}$, where $\overline{G} = G/N = M/N \cdot A/N$, we have $M/N \in \mathbf{F}_\pi$ Since the minimal normal subgroup of M contained in N is normal in M and A, it is also normal in G. Hence N is a minimal normal subgroup of M. Suppose $p \mid |N|$, then N is the p-principal factor of G and M. Therefore $G/C_G(N) \in f(p)$, $C_G(N) = C_{MA}(N) = C_M(N)A$. So

$$\frac{G}{C_G(N)} = \frac{MA}{C_M(N)A} \cong \frac{M}{M \cap C_M(N)A} = \frac{M}{C_M(N)(M \cap A)}.$$

Since $C_M(N) \geq M \cap A$, $G/C_G(N) \cong \dfrac{M}{C_M(N)} \in f(p)$.

Now we come to $M/N \in \mathbf{F}_\pi$, which leads to $M \in \mathbf{F}_\pi$

Because $M = M \cap HA = H(M \cap A)$, $M \in \mathbf{F}_\pi$, $M \cap A \lhd M$, we have $H \in \mathbf{F}_\pi$ by induction on M.

Proof of Theorem 2 (Induction on $|G|$) Assume N is a minimal normal subgroup of G. Put $\overline{G} = G/N$, $\overline{H} = HN/N$. It is obvious that $\overline{H} \lhd\lhd \overline{G}$. By $\Psi_\pi(\overline{G}) = \Phi(\overline{G})O_{\pi'}(\overline{G}) \geq \Phi(G)O_{\pi'}(G)N/N = \Psi_\pi(G)N/N$, we know $\overline{H} \cap \Psi_\pi(\overline{G}) \geq HN/N \cap \Psi_\pi(G)N/N = [H \cap \Psi_\pi(G)]N/N$. Moreover

$$\frac{H}{H \cap \Psi_\pi(G)} \sim \frac{H}{(H \cap \Psi_\pi(G))N} \cong \frac{H/N}{(H \cap \Psi_\pi(G))N/N} \sim \frac{\overline{H}}{\overline{H} \cap \Psi_\pi(\overline{G})} \in \mathbf{F}_\pi$$

Then $\overline{H} = HN/N \cong H/H \cap N \in \mathbf{F}_\pi$ by induction

If G has another minimal normal subgroup N_1, $N \neq N_1$, then $N \cap N_1 = 1$, $H/H \cap N_1 \in \mathbf{F}_\pi$ Hence $H/H \cap N \cap N_1 = H \in \mathbf{F}_\pi$

Now we may suppose that G has unique minimal normal subgroup N, and $N \leq \Psi_\pi(G)$. If N is a π'-group, then $H \in \mathbf{F}_\pi$ Therefore $N \leq \Phi(G)$, N is an elementary abelian p-group, $p \in \pi$ By $H/H \cap N \in \mathbf{F}_\pi$, H is π-solvable.

Now we use induction on the length of composition series of G.

If $H = G$, then $H \in \mathbf{F}_\pi$ by $N \leq \Phi(G)$, $H/N = G/N \in \mathbf{F}_\pi$

1) If $HN > H$, it is obvious that $HN \lhd\lhd G$, and the length of composition series from HN to G is smaller than that of H ([6, P. 133, La 8.6.1]).

By $HN \cap \Psi_\pi(G) = [H \cap \Psi_\pi(G)]N$, we have

$$\frac{H}{H \cap \Psi_\pi(G)} \sim \frac{HN}{[H \cap \Psi_\pi(G))]N} = \frac{HN}{HN \cap \Psi_\pi(G)} \in \mathbf{F}_\pi$$

Then $HN \in \mathbf{F}_\pi$ by induction, which implies $H \in \mathbf{F}_\pi$ by Lemma 5.

2) If $HN = H$, then $N \leq H$, $H \cap \Psi_\pi(G) = N$.

Assume $H \lhd\lhd H_1 \lhd G$. If $\Psi_\pi(H_1) \neq 1$, then $N \leq \Psi_\pi(H_1)$ by $\Psi_\pi(H_1) \lhd G$ and uniqueness of N. So $N \leq H \cap \Psi_\pi(H_1) \leq H \cap \Psi_\pi(G) = N$.

Since the length of composition series from H to H_1 is smaller than that from H to G, $H \in \mathbf{F}_\pi$ by induction.

If $\Psi_\pi(H_1) = 1$, then $\Psi_\pi(H) = 1$. By [4, III, §4, Le 4.4], N has compliments in H and H_1 respectively, $H = MN$, $H_1 = M_1N$. Since H is π-solvable, H_1 has a normal π-solvable subgroup K, $K \geq H$ by Lemma 3. By Lemma 4, the compliment of N in H_1 is conjugate to M_1. By Frattini argument, $G = N_G(M_1)N = N_G(M_1)$, which implies $M_1 \lhd H_1$, $H_1 = M_1 \times N$. Hence $H = M \times N$. But $M \cong M \times N/N = H/N \in \mathbf{F}_\pi$, $N \in \mathbf{F}_\pi$. Therefore $H \in \mathbf{F}_\pi$. Till now, we have proved the Theorem 2.

References

[1] Marian Deaconescu, *Frattini-Like subgroups of Finite Groups*, Math. Rept. Vol. 2, Part 4, Harwood Acad. Publ. GmbH, Great Britain, 1985.

[2] Michael Weinstein, *Between Nilpotent and Solvable*, Polygonal Publ. House, USA, 1982.

[3] Mukherjee, *On the intersection of a class of maximal of finite groups*, J. Pure Appl. Alg., **42**: **2**(1986), 117- 124.

[4] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin Heidelberg New York, 1967.

[5] John S. Rose, *A Course on Group Theory*, Camb. Univ. Press, London, New York, Melbouren, 1978.

[6] Jr. Marshall Hall, *The Theory of Groups*, Macmillan Company, New York, 1959.

π-Frattini 子群与 π-局部群系

陈 重 穆

(西南师范大学数学系, 重庆 400715)

摘 要

文中, 对 π-Frattini 子群给出了更精细的结果, 并将 Gaschütz 幂零性定理推广到 π-局部定义群系. 主要结果是: 设 G 为有限群, H 为 G 的次正规子群. 若 $H/H \cap \Phi(G)O_{\pi'}(G) \in \mathbf{F}$, 则 $H \in \mathbf{F}_\pi$, 其中 $\mathbf{F}_\pi$ 是 π-可解 π-局部定义群系.