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A Note on Hölder Continuity of the Gradient of the Solutions for $u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ *

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Abstract In [1- 3], Hölder continuity for the spatial gradient of weak solutions of

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) \text{ in } \Omega_T = \Omega \times (0, T)$$

was established, where $\nabla = \mathrm{grad}_x$, $\Omega \subset R^N$. We discuss here how the condition

$$p > \max\{1, \frac{2N}{N+2}\}$$

is determined by the behaviour of u. Theorem implies that for solution in $L^N_{loc}(\Omega_T)$, the condition for the gradient of solution being Hölder continuous needs only $p > 1$.

Keywords Hölder continuity of the gradient, weak solution.

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1. Introduction

This paper is a supplement of the results in [1- 3]. There Hölder continuity for the spatial gradient of weak solutions of

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) \text{ in } \Omega_T = \Omega \times (0, T) \tag{1.1}$$

was established. Here $\nabla = \mathrm{grad}_x$, $\Omega \subset R^N$. They proved that if

$$p > \max\{1, \frac{2N}{N+2}\}, \tag{1.2}$$

then the gradient of any solution $u \in L^\infty(0, T: L^2(\Omega)) \cap L^p(0, T: W^{1,p}(\Omega))$ of (1.1) is Hölder continuous.

In this note, we discuss how the condition (1.2) is determined by the behaviour of u.

Definition *A solution u of* (1.1) *is a function defined in $\Omega_T = \Omega \times (0, T)$ such that*

$$u \in L^\infty(0, T: L^q(\Omega)) \cap L^p(0, T: W^{1,p}(\Omega)), \quad q \geq 1, \tag{1.3}$$

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and for any $\varphi \in C_0^1(\Omega_T)$, *u satisfies*

$$\iint_{\Omega_T} [u\varphi_t - |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi] dx\,dt = 0. \tag{1.4}$$

Our result is as follows

Theorem *Suppose that*

$$p > \max\{1, \frac{2N}{N+q}\}, \tag{1.5}$$

where q is the constant in (1.3). *Then the gradient of the solution of* (1.1) *is local Hölder continuous in* Ω_T.

When $q = 2$, the Theorem reduces to the results in [1-3].

Moreover, this Theorem implies that for local bounded solutions, the condition for the gradient of solution being Hölder continuous needs only $p > 1$.

2 Proof of Theorem

Lemma 2.1 *Let* $p > \max\{1, \frac{2N}{N+q}\}$. *Then*

$$u \in L^s_{loc}(\Omega_T) \text{ for any } 1 \le s < +\infty. \tag{2.1}$$

Proof If $q = 1$, then (2.1) follows from [4]. Let $q > 1$. In (1.4) take test function

$$\varphi = u v^{\alpha} \xi^p,$$

where $v = u^2$, $\alpha > -\frac{1}{2}$, $\xi \in C_0^{\infty}(\Omega_T)$, $0 \le \xi \le 1$. Then

$$\frac{1}{2(\alpha+1)} \int_{\Omega} v^{\alpha+1} \xi^p (x,t)\, dx + (1+2\alpha) \int \int_{\Omega} |\nabla u|^p v^{\alpha} \xi^p \,dx\,ds$$
$$\le \frac{p}{2(\alpha+1)} \int \int_{\Omega} v^{\alpha+1} \xi^{p-1} |\xi_t| dx\,ds + p \int \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \xi \xi^{p-1} u v^{\alpha} dx\,ds.$$

Dropping the third term on the left hand side and using Hölder inequality in the last term, we get

$$\operatorname{ess\,sup}_{t \in (0,T)} \int_{\Omega} (\xi |u|^{\frac{2\alpha+p}{p}})^{\frac{(2\alpha+2)p}{2\alpha+p}} dx + \int_0^T \int_{\Omega} |\nabla (\xi |u|^{\frac{2\alpha+p}{p}})|^p dx\,dt$$
$$\le c \int_0^T \int_{\Omega} |\nabla \xi|^p |u|^{2\alpha+p} dx\,dt + c \int_0^T \int_{\Omega} |\xi_t| \xi^{p-1} |u|^{2\alpha+2} dx\,dt. \tag{2.2}$$

For $\alpha = \frac{q}{2} - 1$, the right hand side of (2.2) is finite. Therefore by embedding inequality ([5], p62), we get $u \in L^r_{loc}(\Omega_T)$, $r = p - 2 + q(1+\frac{p}{N})$ and $p - 2 + q(1+\frac{p}{N}) > q$ by (1.5).

We can now repeat the process with $2(\alpha+1) = p - 2 + q(1+\frac{p}{N})$ and get

$$u \in L^s_{loc}(\Omega_T), \quad s = p - 2 + 2(\alpha+1)(1+\frac{p}{N}),$$

clearly $p - 2 + 2(\alpha+1)(1+\frac{p}{N}) > p - 2 + q(1+\frac{p}{N})$ by (1.5). By proceeding in this fashion, Lemma 2.1 follows

Using Lemma 2.1 as in [2], we can prove the following lemma.

Lemma 2.2 *Let* $p > \max\{1, \frac{2N}{N+q}\}$. *Then* $|\nabla u| \in L^s_{loc}(\Omega_T)$, $1 \le s < +\infty$.

Lemma 2.3 *Let* $p > \max\{1, \frac{2N}{N+q}\}$. *Then* $|\nabla u| \in L^\infty_{loc}(\Omega_T)$.

Lemma 2.3 is obtained by a simple change of the proof of [3, Theorem 2.1]. For clarity, we give the details of the proof.

Proof of Lemma 2.3 For $P_0(x_0, t_0) \in \Omega_T$, set $B(x_0, R) = \{x \in R^N: |x - x_0| < R\}$, $Q(p_0, R) = B(x_0, R) \times (t_0 - R^2, t_0)$ without loss of generaliy, we assume that u_t, $D_x^2 u$ are in suitable L^q spaces (refer to the explanation in [3]). Clearly it is sufficient to discuss the case $1 < p < 2$.

Differentiating (1.1) with respect to x_j, we get

$$\frac{\partial}{\partial t} u_{x_j} - \operatorname{div}(|\nabla u|^{p-2} \nabla u_{x_j} + (\frac{\partial}{\partial x_j} |\nabla u|^{p-2}) \nabla u) = 0. \tag{2.3}$$

Set $\varphi = u_{x_j} v^\alpha \xi^2$, where $v = |\nabla u|^2$, $\alpha \ge 0$, ξ is the usual c^1 cut-off function with respect to $Q(P_0, R)$, $Q(P_0, (1-\sigma)R)$ $(0 < \sigma < 1)$. Multiplying (2.3) by φ and integrating over $Q(P_0, R)$, we get (see p345 in [3])

$$\frac{1}{2(\alpha+1)} \operatorname{esssup}_{t_0 - R < t < t_0} \int_{B_R} \xi^2 v^{x+1} dx + \frac{(1+2\alpha)(p-1)}{4} \iint_{Q_R} \xi^2 v^{\frac{p+2\alpha-4}{2}} |\nabla v|^2 dx\,dt$$
$$\le c \iint_{Q(R)} \xi v^{\frac{p+2\alpha-2}{2}} |\nabla \xi| |\nabla v| dx\,dt + \frac{1}{2(\alpha+1)} \iint_{Q(R)} |\xi_t| v^{\alpha+1} dx\,dt,$$

where c is independent of α. Using Hölder inequality and Sobolev's embedding theorem, we get

$$\iint_{Q(R-\sigma R)} v^{\frac{p-2}{2} + (\alpha+1)(1+\frac{\alpha}{N})} dx\,dt$$
$$\le \operatorname{esssup}_{t_0 - (1-\sigma)^2 R^2 < t < t_0} \left(\int_{B(R-\sigma R)} v^{\alpha+1} dx\right)^{2/N} \times \iint_{Q(R-\sigma R)} \left(v^{\frac{p+2\alpha}{2}} + |\nabla v^{\frac{p+2\alpha}{4}}|^2\right) dx\,dt$$
$$\le c\left\{\iint_{Q(R)} v^{\frac{p+2\alpha}{2}} |\nabla \xi|^2 dx\,dt + \iint_{Q(R)} |\xi_t| v^{\alpha+1} dx\,dt\right\}^{1+\frac{\alpha}{N}}. \tag{2.4}$$

Set $k = 1 + \frac{2}{N}$ and $R_l = R(\frac{1}{2} + \frac{1}{2^{l+1}})$, $\alpha + 1 = \frac{N(2-p)}{4} + k^l$, $l = 1, 2, \cdots$. $\xi_l \in C_0^1(Q(R_l))$, $0 \le \xi_l \le 1$, $\xi_l = 1$ on $Q(R_{l+1})$. Then it follows from (2.4) that

$$\left(\iint_{Q(R_{l+1})} v^{\frac{N(2-p)}{4} + k^{l+1}} dx\,dt\right)^{1/k}$$
$$\le c\frac{4^l}{R^2}\left\{\iint_{Q(R_l)} v^{\frac{N(2-p)}{4} + k^l} dx\,dt + \iint_{Q(R_l)} v^{\frac{(N-2)(2-p)}{4} + k^l} dx\,dt\right\}. \tag{2.5}$$

Without loss of generality, we may assume that for any $l > l_0$

$$\iint_{Q(R_l)} v^{\frac{(N-2)(2-p)}{4} + k^l} dx\,dt \ge 1,$$

where l_0 is a natural number such that $\frac{(N-2)(2-p)}{4} + k^{l_0} > 0$.

Hence we get from (2. 5)

$$\left(\iint_{Q(R_{l+1})} v^{\frac{N(2-p)}{4}+k^{l+1}} dx\,dt\right)^{1/k} \leq c\frac{4^l}{R^2}\iint_{Q(R_l)} v^{\frac{N(2-p)}{4}+k^l} dx\,dt.$$

The standard Moser iteration procedure yields

$$\left(\iint_{Q(R_{l+1})} v^{\frac{N(2-p)}{4}+k^{l+1}} dx\,dt\right)^{1/k^{l+1}} \leq \left(\frac{c}{R^{N+2}}\iint_{Q(R)} v^{\frac{N(2-p)}{4}+k^{l_0}} dx\,dt\right)^{k^{-l_0}}.$$

Letting $l \to \infty$, we have

$$\operatorname{esssup}_{Q(R/2)} v \leq \left(\frac{c}{R^{N+2}}\iint_{Q(R)} |\nabla u|^{\frac{N(p-2)}{2}+2k^{l_0}} dx\,dt\right)^{k^{-l_0}},$$

hence by Lemma 2. 2, Lemma 2. 3 is proved.

Using Lemma 2. 3 as in [3], we can prove the Theorem.

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关于 $u_t = \mathbf{div}(|\nabla u|^{p-2}\nabla u)$ 方程解的梯度 Hölder 连续性的一个注记

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摘　要

对文献[1- 3]中的结果: $u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ 在 $\Omega_T = \Omega\times(0, T)$ 上弱解的空间梯度是 Hölder 连续的做一个补充. 在这个注记里, 讨论了条件 $p > \max\{1, \frac{2N}{N+2}\}$ 是怎样由 u 的性质所决定的. 属于 $L^N_{loc}(\Omega_T)$ 空间解的梯度是 Hölder 连续的条件仅仅是 $p > 1$.