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Iterative Construction of Solutions to Nonlinear Equations of Strongly Accretive Operators in Banach Spaces *

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Abstract In this paper, we investigate the Ishikawa iteration process in a p-uniformly smooth Banach space X. Motivated by Deng[6] and Tan and Xu[8], we prove that the Ishikawa iteration process converges strongly to the unique solution of the equation $Tx = f$ when T is a Lipschitzian and strongly accretive operator from X to X, or to the unique fixed point of T when T is a Lipschitzian and strictly pseudo contractive mapping from a bounded closed convex subset C of X into itself. Our results improve and extend Theorem 4.1 and 4.2 of Tan and Xu[8] by removing the restrion

$$\lim_{n\to\infty}\beta_n = 0 \ or \ \lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0$$

in their theorems. These also extend Theorems 1 and 2 of Deng[6] to the p-uniformly smooth Banach space setting.

Keywords strongly accretive, strictly pseudocontractive, p-uniformly smooth Banach space.

Classification AMS(1991) 47H15, 47H05/CCL O177.2

1. Introduction and preliminaries

Let T be a nonlinear operator with domain $D(T)$ and range $R(T)$ in a real Banach space. T is said to be accretive[2] if the inequality

$$\| x - y \| \leq \| x - y + r(Tx - Ty) \| \tag{1}$$

holds for each x and y in $D(T)$ and for all $r \geq 0$. If (1) holds only for some $r > 0$, T is said to be monotone[3]. If X is a Hilbert space then the accretive condition (1) reduces to

$$\langle Tx - Ty, x - y \rangle \geq 0 \tag{2}$$

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for all x, y in X. T is accretive if and only if for any $x, y \in D(T)$, there exists $j \in J(x-y)$ such that $\langle Tx - Ty, j\rangle \geq 0$, where

$$J(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}, \quad x \in X, \tag{3}$$

is the normalized duality mapping of X and $\langle \cdot, \cdot\rangle$ denotes the duality pairing between X and X^*. The accretive operators were introduced independently by Browder[2] and Kato[3] in 1967.

Let C be a nonempty subset of a real Banach space X. A mapping $T: C \to X$ is said to be strongly accretive if for each x, y in C there is $j \in J(x-y)$ such that

$$\langle Tx - Ty, j\rangle \geq k\|x - y\|^2 \tag{4}$$

for some real constant $k > 0$. Without loss of generality, we assume that $k \in (0,1)$.

Let C be a nonempty subset of a real Banach space x. A mapping $T: C \to X$ is said to be srictly pseudocontractive if there exists $t > 1$ such that the inequality

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\| \tag{5}$$

holds for all x, y in C and $r > 0$. If, in the above definition, $t = 1$, then T is said to be a pseudocontractive mapping.

Recently, Deng[6] answered positively Problem 2 in Chidume[7], by removing the restriction $\alpha_n \leq \beta_n$ and $\lim_n \beta_n = 0$. On the other hand, Tan and Xu[8] studied both the Mann and the Ishikawa iteration process in a p-uniformly smooth Banach space X and proved that the two processes converge strongly to the unique solution of the equation $Tx = f$ in case T is a Lipschitzian and strongly accretive operator from X to X, or to the unique fixed point of T in case T is a Lipschitzian and strictly pseudocontractive mapping from a bounded closed convex subset C of X into itself. Hence, Tan and Xu[8] gave affirmative answers to Problems 1 and 2 of Chidume[7] respectively, and also extended all results of Chidume[7] to the p-uniformly smooth Banach space setting.

In this paper, we investigate the Ishikawa iteration process in a p-uniformly smooth Banach space X. Our results improve and extend Theorem 1 and 2 of Deng[6] and Theorem 4.1 and 4.2 of Tan and Xu[8].

To procced, we give some preliminaries. Let X be a Banach space. Recall that the modulus $\rho_x(\cdot)$ of smoothness of X is defined by

$$\rho_x(\tau) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x, y \in X, \|x\| = 1, \|y\| = \tau\},$$

$\tau > 0$, and that X is said to be uniformly smooth if $\lim_{\tau \downarrow 0} \rho_x(\tau)/\tau = 0$. Recall also that for a real number $1 < p \leq 2$, a Banach space X is said to be p-uniformly smooth if $\rho_x(\tau) \leq d\tau^p$ for $\tau > 0$ where $d > 0$ is a constant. It is known (cf. [1]) that for a Hilbert space H,

$$\rho_H(\tau) = (1 + \tau^2)^{1/2} - 1 \tag{6}$$

and hence H is 2-uniformly smooth. It is also known that if $1 < p < 2$, L^p (or l^p) is p-uniformly smooth; while if $2 \leq p < \infty$, L^p (or l^p) is 2-uniformly smooth. Xu[5] gave the following characterization for a p-uniformly smooth Banach space.

Lemma 1 *Let* X *be a smooth Banach space and p a fixed number in* (1,2]. *Then*, X *is p- uniformly smooth if and only if there exists a constant* $d_p > 0$ *such that*

$$\| x + y \|^p \leq \| x \|^p + p\langle y, J_p(x)\rangle + d_p \| y \|^p \tag{7}$$

for all x, y *in* X, *where* $J_p(x)$ *is the subdifferentiable at* x *of the functional* $p^{-1} \| \cdot \|^p$.

It is known that $J_p(x) = \| x \|^{p-2} J(x)$ for $x \in X$, $x \neq 0$, and

$$J_p(x) = \{ x^* \in X^* : \langle x, x^* \rangle = \| x \|^p, \| x^* \| = \| x \|^{p-1} \}, \quad x \in X.$$

When x is an L^p (or l^p) space, the constant d_p in (1.7) has been calculated.

2. Main results

Theorem 1 *let* X *be a* p- *uniformly smooth Banach space with* $1 < p \leq 2$ *and* $T: X \to X$ *be a Lipschitzian and strongly accretive operator with Lipschitz constant* L. *Define* $S: X \to X$ *by* $Sx = f - Tx + x$. *Let* $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$ *be two sequences of reals in* [0,1] *satisfying*

(i) $\sum_{n=0}^{\infty} a_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $0 \leq \beta_n \leq \min\{ t_p, \frac{k}{4p^2 L_0 (1 + L_0^p)^{1/p}} \}$ *for each* $n \geq 0$,

where L_0 *is the Lipschitz constant of S with* $L_0 \leq 1 + L$, t_p *is the (smaller) solution of the equation*

$$f(t) = p(p-1)(1-k)t - (1 + d_p L_0^p) t^{p-1} + \frac{1}{2} pk = 0 \ (t > 0), \tag{8}$$

and $k \in (0,1)$, d_p are the constants appearing in (4) and (7), respectively. Then for each x_0 in X the Ishikawa sequence $\{x_n\}$ defined by

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n S y_n \text{ and } y_n = (1 - \beta_n) x_n + \beta_n S x_n, \quad n \geq 0$$

converges strongly to the unique solution of the equation $Tx = f$.

Proof We first observe that the equation $Tx = f$ has a unique solution which we denote by q. In fact, the existence follows from Morales[4] and the uniqueness from the strong accretiveness of T. We also observe that for $x, y \in X$,

$$\begin{aligned}\langle Sx - Sy, J_p(x-y)\rangle &= -\langle Tx - Ty, J_p(x-y)\rangle + \| x - y \|^p \\ &= -\| x - y \|^{p-2}\langle Tx - Ty, J(x-y)\rangle + \| x - y \|^p \\ &\leq -k \| x - y \|^{p-2} \| x - y \|^2 + \| x - y \|^p \\ &= (1-k) \| x - y \|^p.\end{aligned}$$

It follows that

$$\begin{aligned}\| x_{n+1} - q \|^p &= \| (1-\alpha_n)(x_n - q) + \alpha_n (S y_n - q) \|^p \\ &\leq (1-\alpha_n)^p \| x_n - q \|^p + p\alpha_n (1-\alpha_n)^{p-1}\langle S y_n - q, J_p(x - q)\rangle \\ &\quad + d_p \alpha_n^p \| S y_n - q \|^p.\end{aligned} \tag{9}$$

Since

$$\|Sy_n - q\|^p \le L_0^p \|y_n - q\|^p$$
$$\langle Sx_n - q, J_p(x_n - q)\rangle \le (1 - k)\|x_n - q\|^p,$$

$$\begin{aligned}\|y_n - q\|^p &= \|(1-\beta_n)(x_n - q) + \beta_n(Sx_n - q)\|^p \\ &\le (1-\beta_n)^p\|x_n - q\|^p + p\beta_n(1-\beta_n)^{p-1}\langle Sx_n - q, J_p(x_n - q)\rangle \\ &\quad + d_p\beta_n^p\|Sx_n - q\|^p \\ &\le ((1-\beta_n)^p + p(1-k)\beta_n(1-\beta_n)^{p-1} + d_pL_0^p\beta_n^p)\|x_n - q\|^p \\ &= t_n\|x_n - q\|^p,\end{aligned}$$

where $t_n = (1-\beta_n)^p + p(1-k)\beta_n(1-\beta_n)^{p-1} + d_pL_0^p\beta_n^p$,

$$\begin{aligned}\|y_n - x_n\|^p = \beta_n^p\|x_n - Sx_n\|^p &= \beta_n^p\|(x_n - q) + (q - Sx_n)\|^p \\ &\le 2^p\beta_n^p(\|x_n - q\|^p + \|Sx_n - q\|^p) \\ &\le 2^p(1 + L_0^p)\beta_n^p\|x_n - q\|^p,\end{aligned}$$

$$\begin{aligned}\langle Sy_n - Sx_n, J_p(x_n - q)\rangle &\le L_0\|y_n - x_n\|\cdot\|x_n - q\|^{p-1} \\ &\le 2L_0\beta_n(1 + L_0^p)^{1/p}\|x_n - q\|^p,\end{aligned}$$

and

$$\begin{aligned}\langle Sy_n - q, J_p(x_n - q)\rangle &= \langle Sy_n - Sx_n, J_p(x_n - q)\rangle + \langle Sx_n - q, J_p(x_n - q)\rangle \\ &\le (2L_0\beta_n(1 + L_0^p)^{1/p} + (1-k))\|x_n - q\|^p,\end{aligned}$$

we obtain from (8)

$$\begin{aligned}\|x_{n+1} - q\|^p \le ((1-\alpha_n)^p &+ p\alpha_n(1-\alpha_n)^{p-1}(1 - k + 2L_0\beta_n(1 + L_0^p)^{1/p}) \\ &+ d_pL_0^p\alpha_n^p t_n)\|x_n - q\|^p.\end{aligned}$$

Since $1 < p \le 2$, $(1-t)^p \le 1 - pt + t^p$ and $(1-t)^{p-1} \le 1 - (p-1)t$ for $0 \le t \le 1$, we obtain

$$\begin{aligned}t_n &= (1-\beta_n)^p + p(1-k)\beta_n(1-\beta_n)^{p-1} + d_pL_0^p\beta_n^p \\ &\le 1 - pk\beta_n - p(p-1)(1-k)\beta_n^2 + (1 + d_pL_0^p)\beta_n^p. \qquad (10)\end{aligned}$$

Since $\beta_n \le t_p$ for all $n \ge 0$, we have from (8) $p(p-1)(1-k)\beta_n^2 - (1 + d_pL_0^p)\beta_n^p \ge -\frac{1}{2}pk\beta_n$. Hence it follows that $t_n \le 1 - \frac{1}{2}pk\beta_n$ for each $n \ge 0$. On the other hand, since $\lim_{n\to\infty}\alpha_n = 0$, there exists a positive integer N such that $0 \le \alpha_n \le t_p$ for each $n \ge N$. This implies that

$$\begin{aligned}t'_n &= (1-\alpha_n)^p + p(1-k)\alpha_n(1-\alpha_n)^{p-1} + d_pL_0^p\alpha_n^p \\ &\le 1 - \frac{1}{2}pk\alpha_n \text{ for each } n \ge N.\end{aligned}$$

Therefore, we obtain that for each $n \geq N$,

$$\begin{aligned}
&\| x_{n+1} - q \|^p \\
&\leq [(1-\alpha_n)^p + p\alpha_n(1-\alpha_n)^{p-1}(1-k) + p\alpha_n(1-\alpha_n)^{p-1} \cdot 2L_0\beta_n(1+L_0^p)^{1/p} \\
&\quad + d_p L_0^p \alpha_n^p (1 - \frac{1}{2}pk\beta_n)] \| x_n - q \|^p \\
&\leq [t_n' + p(\alpha_n - (p-1)\alpha_n^2) \cdot 2L_0\beta_n(1+L_0^p)^{1/p} - \frac{1}{2}pkd_pL_0^p\alpha_n^p\beta_n] \| x_n - q \|^p \\
&\leq [1 - \frac{1}{2}pk\alpha_n + 2pL_0(1+L_0^p)^{1/p}\alpha_n\beta_n - 2p(p-1)L_0(1+L_0^p)^{1/p}\alpha_n^2\beta_n \\
&\quad - \frac{1}{2}pkd_pL_0^p\alpha_n^p\beta_n] \| x_n - q \|^p.
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$ implies $\lim_{n\to\infty}\alpha_n\beta_n = 0$, we have

$$\begin{aligned}
&\lim_{n\to\infty}[\alpha_n\beta_n]^{-1}\{2pL_0(1+L_0^p)^{1/p}\alpha_n\beta_n - 2p(p-1)L_0(1+L_0^p)^{1/p}\alpha_n^2\beta_n - \frac{1}{2}pkd_pL_0^p\alpha_n^p\beta_n\} \\
&= 2pL_0(1+L_0^p)^{1/p} < 2p^2L_0(1+L_p^p)^{1/p}.
\end{aligned}$$

From this and the condition (ii), we derive that there is a positive integer $N_0 > N$ *such that for each* $n \geq N_0$,

$$\begin{aligned}
&\| x_{n+1} - q \|^p \\
&\leq [1 - \frac{1}{2}pk\alpha_n + 2p^2L_0(1+L_0^p)^{1/p}\alpha_n\beta_n] \cdot \| x_n - q \|^p \\
&\leq [1 - \frac{1}{2}pk\alpha_n + 2p^2L_0(1+L_0^p)^{1/p} \cdot \frac{k}{4p^2L_0(1+L_0^p)^{1/p}} \cdot \alpha_n] \| x_n - q \|^p \\
&\leq [1 - \frac{1}{2}(p-1)k\alpha_n] \| x_n - q \|^p \\
&\leq [\exp(-\frac{1}{2}(p-1)k\alpha_n)] \| x_n - q \|^p \\
&\leq [\exp(-\frac{1}{2}(p-1)k \cdot \sum_{j=N_0}^{n}\alpha_j)] \| x_{N_0} - q \|^p.
\end{aligned}$$

This immediately implies the strong convergence of $\{x_n\}$ *to* q *since the series* $\sum_n \alpha_n$ *diverges. The proof is complete.* □

Reviewing the proof of Theorem 1, we can see that the following consequence is true.

Theorem 2 *Let C be a nonempty bounded closed convex subset of a p-uniformly smooth Banach space X with $1 < p \leq 2$ and $T: C \to C$ be a Lipschitzian and strictly pseudocontractive mapping with Lipschitz constant L. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be sequences of reals in $[0,1]$ satisfying*

(i) $\sum_{n=0}^{\infty}\alpha_n = \infty$ *and* $\lim_{n\to\infty}\alpha_n = 0$;

(ii) $0 \le \beta_n \le \min\{t_p, \frac{k}{4p^2L(1+L^p)^{1/p}}\}$ for each $n \ge 0$,

where t_p is the (smaller) solution of the equation

$$f(t) = p(p-1)(1-k)t - (1+d_pL^p)t^{p-1} + \frac{1}{2}pk = 0 \quad (t > 0),$$

$k = (t-1)/t$ and $t \in (1, \infty)$, d_p are the constants appearing in (5) and (7), respectively. Then, for each x_0 in C, the Ishikawa sequence $\{x_n\}$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n \text{ and } y_n = (1-\beta_n)x_n + \beta_n Tx_n, \ n \ge 0$$

converges strongly to the unique fixed point of T.

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Banach 空间中强增生算子的非线性方程的解的迭代构造

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摘　要

本文研究 p 一致光滑 Banach 空间 X 中 Ishikawa 迭代法. 受 Deng[6] 与 Tan, Xu[8] 的启发，证明了，当 T 是从 X 到自身的 Lipschitz 强增生算子时，Ishikawa 迭代法强收敛到方程 $Tx = f$ 的唯一解；当 T 是从 X 的有界闭凸子集到自身的 Lipschitz 严格伪压缩映象时，Ishikawa 迭代法强收敛到 T 的唯一不动点. 通过去掉限制 $\lim_{n\to\infty}\beta_n = 0$ 或 $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0$，结果改进与推广了 Tan, Xu[8] 的定理 4.1 与定理 4.2，也把 Deng[6] 的定理 1 与定理 2 推广到了 p 一致光滑 Banach 空间的背景.