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On Unicity of Meromorphic Functions and Their Derivatives *

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Abstract In this paper, we prove the following theorem: Suppose that k, n be two positive integers, and a, b, w be three finite complex numbers with $a^n \neq b^n$, $w^n = 1$. If a meromorphic function $f(z)$ and its k - th derivative $f^{(k)}(z)$ share two finite set $S_1 = \{aw^i \mid i = 1, 2, \cdots, n\}$, $S_2 = \{bw^i \mid i = 1, 2, \cdots, n\}$, then $f(z) = tf^{(k)}(z)$, where $t^n = 1$.

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1. Introduction

We say that two non - constant meromorphic functions $f(z)$ and $g(z)$ on C share the value a if the zeros of $f(z) - a$ and the zeros of $g(z) - a$ ($\frac{1}{f}$ and $\frac{1}{g}$, if $a = \infty$) are the same (counting multiplicity). In this paper the term "meromorphic" will always mean "meromorphic in the whose complex plane". Rubel and Yang[1] proved the following theorem.

Theorem A *If an non - constant entire function $f(z)$ and its derivative $f'(z)$ share two finite complex values, then $f(z) \equiv f'(z)$.*

Gundersen[2], Frank and Ohlenroth[3], Frank and Weissenborn[4] extended Theorem A by proving

Theorem B *If an non - constant entire function $f(z)$ and its k - th derivative $f^{(k)}(z)$ share two finite complex values, then $f(z) \equiv f^{(k)}(z)$.*

In order to state our result, we introduce the following definition.

Definition 1 *Suppose that $a_1, a_2, \cdots, a_n$ be finite complex numbers. We say that $f(z)$ and $g(z)$ share a finite complex set $S = \{a_i \mid i = 1, 2, \cdots, n\}$ if and only if the zreos of $\prod_{i=1}^{n}(f(z) - a_i)$ and the zeros of $\prod_{i=1}^{n}(g(z) - a_i)$ are the same (counting multiplicity).*

In this paper, we prove the following theorem.

Theorem 1 *Suppose that k, n be two positive integers, and a, b, w be three finite complex*

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numbers with $a^n \neq b^n$, $w^n = 1$. *If a meromorphic function* $f(z)$ *and its k-th derivative* $f^{(k)}(z)$ *share two finite set* $S_1 = \{aw^i \mid i = 1, 2, \cdots, n\}$, $S_2 = \{bw^i \mid i = 1, 2, \cdots, n\}$ *then* $f(z) = tf^{(k)}(z)$, *where* $t^n = 1$.

2. Same Lemmas

From the proof of Theorem 1 in [5], we can obtain the following Lemma.

Lemma 1 *Assume that* $f(z)$ *and* $g(z)$ *are non-constant meromorphic functions such that*

$$\left.\begin{aligned} N(r, \frac{1}{f}) + N(r, f) &= S(r, f) \\ N(r, \frac{1}{g}) + N(r, g) &= S(r, g) \end{aligned}\right\} \quad ($$

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On the other hand, we have

$$N(r,1) - \overline{N}(r,1) \leq S(r,f) + S(r,g). \tag{6}$$

To prove (6), set

$$\psi(z) = \frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}.$$

If $\psi(z) \not\equiv 0$, then

$$\begin{aligned} N(r,1) - \overline{N}(r,1) &\leq N\left(r,\frac{1}{\psi}\right) \\ &\leq T(r,\psi) + O(1) \\ &= N(r,\psi) + m(r,\psi) + O(1) \\ &\leq S(r,f) + S(r,g). \end{aligned}$$

So we have $\psi(z) \equiv 0$, that is

$$\frac{f'(z)}{f(z)} \equiv \frac{g'(z)}{g(z)}.$$

Hence $f(z) \equiv cg(z)$, where c is a constant, which contradicts the hypothesis of Lemma 2. From (5) and (6), we deduce

$$N(r,1) \leq \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{g}\right) + S(r,f) + S(r,g).$$

Obviously, from (1), we have

$$N(r,0) \leq \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{g}\right) + S(r,f) + S(r,g).$$

3. Proof of Theorem 1

From Theorem B, we can assume that $n \geq 2$.

Case 1 Assume that $N(r,f) = S(r,f)$, then from the hypothesis of Theorem 1, we obtain $S(r,f) = S(r,f^{(k)})$, hence we have $N(r,f^{(k)}) = S(r,f^{(k)})$. Set $F(z) = f^{(k)}(z)$, $G(z) = (f^{(k)}(z))^n$. Obviously, $S(r,F) = S(r,G)$. For simplicity of notion we denote $S(r,F)$ by $S(r)$. Hence, we have $N(r,F) = N(r,G) = S(r)$, thus $F(z)$ and $G(z)$ share the values a^n, b^n. Without loss of generality, we assume that $a^n = 0$, $b^n = 1$.

If $F(z)$ is not a Möbius transformation of $G(z)$, then by Lemma 2

$$N(r,1) \leq \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{G}\right) + S(r), \tag{7}$$

$$N(r,0) \leq \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{G}\right) + S(r). \tag{8}$$

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By the second fundamental theorem, we have

$$T(r,F) \le N(r,0) + N(r,1) - N(r,\frac{1}{F}) + S(r), \tag{9}$$

$$T(r,G) \le N(r,0) + N(r,1) - N(r,\frac{1}{G}) + S(r). \tag{10}$$

Hence from (7) - (10), we obtain

$$2\{T(r,F) + T(r,G)\} \le 3\{N(r,0) + N(r,1)\} + S(r). \tag{11}$$

On the other hand,

$$\begin{aligned} T(r,F) &\le N(r,F) + N(r,0) + N(r,1) + S(r) \\ &\le N(r,\frac{1}{F(z)-G(z)}) + S(r) \\ &\le T(r,F(z)-G(z)) + S(r) \le m(r,F(z)-G(z)) + S(r) \\ &\le m(r,F(z)) + S(r) \le T(r,F) + S(r). \end{aligned}$$

So we have

$$N(r,0) + N(r,1) = T(r,F) + S(r). \tag{12}$$

Since

$$\begin{aligned} 2T(r,F) &= T(r,\frac{1}{F}) + T(r,\frac{1}{F-1}) + S(r) \\ &= N(r,0) + N(r,1) + m(r,\frac{1}{F}) + m(r,\frac{1}{F-1}) + S(r) \\ &\le T(r,F) + m(r,\frac{1}{F}) + m(r,\frac{1}{F-1}) + S(r). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} T(r,F) &\le m(r,\frac{1}{F}) + m(r,\frac{1}{F-1}) + S(r) \\ &\le m(r,\frac{1}{F} + \frac{1}{F-1}) + S(r) \\ &\le m(r,\frac{1}{G}) + S(r) \le T(r,G) + S(r). \end{aligned}$$

On the other hand,

$$T(r,G) \le T(r,F) + S(r).$$

So we have

$$T(r,F) = T(r,G) + S(r). \tag{13}$$

Combining (11), (12) and (13), we deduce that $T(r,F) \le S(r)$, a contradiction.

Hence, $F(z)$ is a Möbius transformation of $G(z)$. Let

$$F(z) \equiv \frac{AG(z)+B}{CG(z)+D}, \tag{14}$$

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where A, B, C and D are finite complex numbers satisfying $AD - BC \neq 0$.

(1.1) If $N(r,0) = S(r)$, then by Lemma 1, we obtain that either $F(z) \equiv G(z)$ or $F(z)G(z) \equiv 1$. If $F(z)G(z) \equiv 1$, that is $f^n(z)(f^{(k)}(z))^n \equiv 1$, then combining (14), we can deduce a contradiction. Thus, we have $F(z) \equiv G(z)$.

(1.2) If $N(r,1) = S(r)$, similarly, we have $F(z) \equiv G(z)$.

(1.3) If $N(r,0) \neq S(r)$, $N(r,1) \neq S(r)$, then by (14), we have

$$F(z) \equiv \frac{(C+D)G(z)}{CG(z)+D}. \tag{15}$$

If $C \neq 0$, then

$$N\left(r, \frac{1}{G(z)+\frac{D}{C}}\right) = N(r,F) = S(r),$$

$$\begin{aligned} 2T(r,G) &\leq N(r,0) + N(r,1) + N\left(r, \frac{1}{G(z)+\frac{D}{C}}\right) + S(r) \\ &\leq N(r,0) + N(r,1) + S(r). \end{aligned}$$

Hence, by (12), we deduce $T(r,G) = S(r)$. Therefore, $F(z) \equiv G(z)$.

Case 2 Set

$$H(z) = \frac{F(z)(G(z)-1)}{G(z)(F(z)-1)}. \tag{16}$$

Obviously,

$$N(r,H) + N\left(r,\frac{1}{H}\right) + S(r). \tag{17}$$

By (16), we have

$$G - F = (H-1)G(F-1). \tag{18}$$

Let z_0 is the pole of $F(z)$, then by (18), we deduce that z_0 is a zero of multiplicity $l(\geq 2)$ of $H-1$. If $H'(z) \not\equiv 0$, then

$$\begin{aligned} N(r,F) &= N_{(2}(r,F) \\ &\leq 2N\left(r,\frac{1}{H}\right) + S(r) \\ &\leq 2N\left(r,\frac{H}{H}\right) + 2N\left(r,\frac{1}{H}\right) + S(r) \\ &\leq 2N\left(r,\frac{H}{H}\right) + S(r) = 2T\left(r,\frac{H}{H}\right) + S(r) \\ &= 2N\left(r,\frac{H}{H}\right) + 2m\left(r.\frac{H}{H}\right) + S(r), \end{aligned} \tag{19}$$

that is $N(r,f) = S(r)$. Hence, by Case 1 we obtain $F(z) \equiv G(z)$.

If $H'(z) \equiv 0$, then $H(z) \equiv C$, obviously C is a non-zero constant. If $C = 1$, then $F(z) \equiv G(z)$. If $C \neq 1$, then $N(r,F) = S(r)$. Hence, from Case 1 we have $F(z) \equiv G(z)$.

Therefore, $f(z) \equiv tf^{(k)}(z)$, where $t^n = 1$.

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亚纯函数及其导数的唯一性

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摘 要

本文证明了如下结果:设 k, n 是两个正整数, a, b, w 是三个有穷复数,满足 $a^n \neq b^n$, $w^n = 1$. 如果一开平面上的亚纯函数 $f(z)$ 以及它的 k 阶导数 $f^{(k)}(z)$ 分担两个集合 $S_1 = \{aw^i \mid i = 1, 2, \cdots, n\}$, $S_2 = \{bw^i \mid i = 1, 2, \cdots, n\}$,则 $f(z) \equiv tf^{(k)}(z)$,其中 $t^n = 1$.

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