Journal of Mathematical Research & Exposition
Vol. 18, No. 3, 359 - 365, August 1998

The Singular Integral of an Analytic Polyhedron *

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Abstract Combining R. Harvey and J. Porking's methods and traditional methods, we define the current Cauchy principal values in this paper by using homotopy formula and integral transformations. We study the boundary value of Weil type polyhedron integrals and obtain Plemelj formulas, which are different from the methods usually in the studies of boundary value problems.

Keywords singular integral, analytic polyhedron.

Classification AMS(1991) 32A/CCL O174.56

1. Definitions and lemmas

Definition 1[1] *The domain* $\triangle$ *is called the non-degenerate Weil polyhedron in* C^n, *if for a domain* $D \supset \bar{\triangle}$ *there are functions* $F_j(z)$, $j = 1, \cdots, N$, *holomorphic in* D, *such that*

$$\triangle = \{ z \in D : | F_j(z) | < 1, j = 1, \cdots, N \}, \quad n \leq N, \quad D \subset C^n$$

and for

$$z \in S_{j_1 \cdots j_n} = \{ z \in D : | F_{j_1}(z) | = \cdots = | F_{j_k}(z) | = 1 \}$$

$\mathrm{rank}(\mathrm{grad}\, F_{j_1}(z), \cdots, \mathrm{grad}\, F_{j_k}(z))^T = k$, *where* $1 \leq j_1 < \cdots < j_k \leq N$, *and* $1 \leq k \leq n$, *namely* $d\rho_{j1} F_{j1} \wedge \cdots \wedge d\rho_{jk} F_{ji} \neq 0$, ρ_j *being strongly polysubharmonic functions of the neighbourhood in* S_j, $S_j = \{ z \in D, \rho_j(z) = 0, j = 1, \cdots, N \}$.

Lemma 1[2] *Let* $\triangle$ *be the non-degenerate Weil polyhedron over a bounded domain in* C^n *with* $c^{(3)}$ *-boundary. Then for any* $f(z) \in A(\bar{\triangle})$,

$$\int_{\partial} f(\zeta, z) = \frac{(n-1)!}{(2\pi i)^n} \sum_{j_1 < \cdots < j_n} \int_{j_1 \cdots j_n} f(\zeta) D_{j_1 \cdots j_n}$$

$$= \begin{cases} f(z) & \text{if } z \in \triangle, \\ 0 & \text{if } z \in (D - \bar{\triangle}) - \bigcup_{j=1}^{N} \Gamma_j, \quad io \end{cases}$$

where

$$D_{j_1\cdots j_n} = \begin{vmatrix} q_{j_1}1 & \cdots & q_{j_1}n \\ \cdots & \cdots & \cdots \\ q_{j_n}1 & \cdots & q_{j_n}n \end{vmatrix}, \quad q_{ji}(\zeta, z) = \frac{p_{ji}(\zeta, z)}{F_j(\zeta, z)}, \quad j = 1, \cdots, N, \quad i = 1, \cdots, n$$

and $p_{ji}(\zeta, z)$ is determined by Hefer theorem, namely

$$F_j(\zeta, z) = \sum_{i=1}^{n}(\zeta_i - z_i)\, p_{ji}(\zeta, z)$$

and for fixed ζ, $F_j(\zeta, z)$ holomorphic for z, and for fixed z, $F_j(\zeta, z)$ is continuous and differentiable. The other notions are referred to [3].

Because $S_j = \{\zeta \in \alpha\triangle, \rho_j(\zeta) = 0\}$, $j = 1, \cdots, N$, $S_{j_1}, \cdots, j_k = S_{j_1} \cap \cdots \cap S_{j_k}$, the characteristic manifold of dimension n is

$$\Omega = \bigcup_{j_1 < \cdots < j_n} S_{j_1 \cdots j_n}.$$

In addition, it will be required that the defining functions ρ_j of $\triangle$ are real, specifically we will assume that ρ_j has the expansion

$$\rho_j(z) = \rho_j(\zeta) + Re[\mu(\zeta, z)(p_j(\zeta, z)) \cdot (\zeta - z)] + \gamma(\zeta, z) \tag{2}$$

for z and ζ near $\triangle$, where

$$\mu(\zeta, z) \text{ never vanishes and is holomorphic in } z \text{ for each fixed } \zeta, \tag{3}$$

$$|\gamma(\zeta, z)| \leq c|\zeta - z|^2, \tag{4}$$

$$\gamma(\zeta, z) \geq c|\zeta - z|^2 \text{ if } p_j(\zeta, z) \cdot (\zeta - z) = 0. \tag{5}$$

$\gamma(\zeta, z)$ vanishes to second order on the diagonal. Let $Q(z, \zeta - z)$ denote the quadratic part of $\gamma(\zeta, z)$ expanded in ζ about $\zeta = z$ with fixed z. Then

$$\gamma(\zeta, z) = Q(z, \zeta - z) + O(|\zeta - z|^3). \tag{6}$$

Definition 2 *The analytic polyhedrons satisfying the conditions (2) - (6) are called Weil type polyhedrons.*

Lemma 2[4] *Let p_j be generating functions defined on $V \subset C^n \times C^n$. Let $A = \{(\zeta, z) \in V \mid \prod_{j=1}^{N} |p_j(\zeta, z) \cdot (\zeta - z)| = 0\}$ and $\chi_\varepsilon(\zeta, z)$* be the characteristic function of the set

$$\{(\zeta, z) \in V \mid \prod_{j=1}^{N} |p_j(\zeta, z) \cdot (\zeta - z)| \geq \varepsilon\}, \tag{7}$$

where $p_j(\zeta, z) \cdot (\zeta - z) = \sum_{i=1}^{n} p_{ji}(\zeta, z) \cdot (\zeta_i - z_i)$, $j = 1, \cdots, N$, the form K, which smooth in V - A, defines the Cauchy principal value currents on V then for $f(z) \in A_c(\triangle)$ the formulas

$$K_f = \lim_{\varepsilon \to 0} \int_{\{\zeta \mid \prod_{j=1}^{N} p_j(\zeta, z) \cdot (\zeta - z) \mid \geq \varepsilon\} \cap \Omega} K(\zeta, z) \wedge f(\zeta) \tag{8}$$

exist, where U_1 and U_2 are open sets in C^n and V is an open set in $C^n \times C^n$.

2. Plemelj formuls

Theorem *Suppose that $p_j(\zeta, z)$ are support functions for Ω related to $\triangle^- = \{z \in \triangle \mid \prod_{j=1}^N p_j(z) < 0\}$ which satisfies (2) - (6), each coefficient of $k(\zeta, z)$ has a locally integral restriction in Ω with either z or ζ fixed. Then for $f \in C_0^\infty(\Omega)$ and $z \in \Omega$, the principal value integral (8) exists. Futhermore we have Plemelj formula*

$$k^- f(z) = k_b f(z) - \beta(z) f(z), \tag{9}$$

where $\beta(z)$ is a function of z which is independent of ε.

Proof Let χ^+ be the characteristic function of

$$\triangle^+ = \{z \mid \prod_{j=1}^N \rho_j(z) > 0\} \cap (z \in (D - \triangle - \prod_{j=1}^N \Gamma_j).$$

Let $B_\varepsilon = \{\zeta \mid \prod_{j=1}^N \mid p_j(\zeta, z) \cdot (\zeta - z) \mid \le \varepsilon\}$ and χ_ε^+ be the characteristic function of $\triangle - B_\varepsilon$. Then we have

$$\bar\partial \chi = [\Omega]^{0,1}, \bar\partial \chi_\varepsilon^+ = [\Omega_\varepsilon]^{0,1} + [S_\varepsilon]^{0,1},$$

where $S_\varepsilon = \partial B_\varepsilon \cap \triangle^+$ and $\Omega_\varepsilon = \Omega - B_\varepsilon$ is characteristic manifold of dimension n.

By Stokes theorem, we have $d\chi_\triangle = -[\partial \triangle]$, if $\chi^\pm$ are the characteristic functions of $\triangle^-$ and Ω is the oriented characteristic manifold of $\triangle^\pm$, then $d\chi^+ = -d\chi^- = [\Omega]$ in $\triangle$ and $[\Omega] = d\chi^+ = \bar\partial \chi^+ + \partial\chi^+ = [\Omega]^{0,1} + [\Omega]^{1,0}$, $[\Omega]\, d\chi^+ = \sigma d\rho$, where ρ is a strongly polysubharmonic function. If $|d\rho| \equiv 1$ on Ω, then $[\Omega]^{1,0} = \partial\chi^+ = \sigma\, d\rho$, $[\Omega]^{0,1} = \bar\partial \chi^+ = \sigma \bar\partial \rho$.

From (2) - (6), we can choose sufficiently small $\varepsilon > 0$ such that $B_\varepsilon \cap \triangle^+ \subset\subset D$. Let $f(z) \in C_0^\infty(\Omega)$ and $f \equiv 1$ in $B_\varepsilon \cap \triangle^+$. By using homotopy formula for $\chi^\pm f$ and the proof of Theorem 8.38 in [5], for $w \in \triangle^-$ we have

$$k(f[\Omega]^{0,1})(w) = k(f[\Omega_\varepsilon]^{0,1})(w) + k([S_\varepsilon]^{0,1})(w).$$

The right hand side is continuous in $\triangle^- \cap B_\varepsilon$. Thus

$$k^-[f(z)] = k(f[\Omega_\varepsilon]^{0,1})(z) + k([S_\varepsilon]^{0,1})(z) = I_1 + I_2.$$

By Lemma 2 the integral I_1 exists. Therefore it is sufficient to prove

$$\lim_{\varepsilon \to 0^+} k([S_\varepsilon]^{0,1}(z) = -\beta(z).$$

Let

$$\beta_\varepsilon(z) = -k([S_\varepsilon]^{0,1})(z) = \int_{\in S_\varepsilon} k(\zeta, z). \tag{10}$$

By calculating the integral on the right hand side of (10), we have

$$k(\zeta, z) = \frac{(n-1)!}{(2\pi i)^n} D'_{j_1 \cdots j_n}(\prod_{j=1}^N p_j(\zeta, z) \cdot (\zeta - z)]^{-1} d\zeta, \tag{11}$$

where

$$D'_{j_1\cdots j_n} = \begin{vmatrix} p_{j_1}1 & \cdots & p_{j_1}n \\ \cdots & \cdots & \cdots \\ p_{j_n}1 & \cdots & p_{j_n}n \end{vmatrix}.$$

Notice that the integral is taken over the characteristic manifold and order of singular point of the integral with kernel $k(\zeta, z)$ is proper. For simplicity, we assume $D'_{j_1\cdots j_n} \neq 0$ at z.

$$F_j(\zeta, z) = \sum_{i=1}^{n} \frac{\partial\rho_j(\zeta)}{\partial z_i}\Big|_{z=\zeta}(z_i - \zeta_i) + \frac{1}{2}\sum_{i,k}\frac{\partial\rho_j(\zeta)}{\partial z_i \partial z_k}\Big|_{z=\zeta}(z_i - \zeta_i)(z_k - \zeta_k) \tag{12}$$

for ζ in the neighbourhood $\cup(z,\delta)$, $\delta > \varepsilon$. By Taylor's theorem,

$$\rho_j(z) = \rho_j(\zeta) + 2\mathrm{Re}\sum_i \frac{\partial\rho_j(\zeta)}{\partial z_i}(z_i - \zeta_i) + O(|\zeta - z|^2). \tag{13}$$

By (2) - (4), we have

$$\rho_j(z) = \rho_j(\zeta) + \mathrm{Re}(V_j(\zeta, z)(z - \zeta)) + O(|\zeta - z|^2),\ V_j(\zeta, z) = \mu(\zeta, z)\, p_j(\zeta, z).$$

Thus we have the following estimate in $U(z,\delta)$:

$$|ReF_j(\zeta, z)| \geq \frac{1}{2}(|\rho_j(z)| + M|\zeta - z|^2),\}\ M > 0.$$

We use coordinate transformation in $U(z,\delta)$ which change $\zeta_1, \cdots, \zeta_n$ into the real coordinates $x_1, \cdots, x_{2n}$, let $x_j(\zeta)$ be continuous in ζ and $x_i(z) = 0$, $i = 1, \cdots, 2n$. Let $x_1 = \rho_j(\zeta) - \rho_j(z)$ and $x_2 = \mathrm{Im}\, F_j(\zeta, z)$. We have

$$|ReF_j(\zeta, z)| \geq \frac{1}{2}(|\rho_j(z)| + \alpha(x_2^2 + \cdots + x_2^2 n), \alpha > 0.$$

Let $\eta(\zeta) = \rho_j(\zeta) - \rho_j(z) + \mathrm{Im}\, F(z,\zeta) = x_1 + ix_2$, $\eta_k(\zeta) = x_k + ix_{k+1}$, $k = 3, \cdots, 2n-1$. $\eta(\zeta) = (\eta_1(\zeta), \eta_2(\zeta), \cdots, \eta_{2n-1}(\zeta))$. We have

$$|\zeta - z|^2 \geq \gamma^2(x_2^2 + \cdots + x_{2n}^2).$$

Thus

$$\begin{aligned} |\mathrm{Re}\, F_j(\zeta, z)| &\geq \frac{1}{2}[|\rho_j(z)| + \alpha'(x_2^2 + \cdots + x_{2n}^2)] \\ &\geq \frac{1}{2}\min(1, \alpha')[|\rho_j(z)| + (x_2^2 + \cdots + x_{2n}^2)] \\ &= M_1[|\rho_j(z)| + (x_2^2 + \cdots + x_{2n}^2)]. \end{aligned}$$

Since $x_1 = \rho_j(\zeta) - \rho_j(z)$, $x_2 = \mathrm{Im}\, F(\zeta, z)$. Let $x_3 = \zeta_2 - x_2'$, $x_4 = \eta_2 - y_2'$, $\cdots$, $x_{2n} = \eta_n - y_n'$, $z_\alpha = x_\alpha' + iy_\alpha'$, $\zeta_\alpha = \zeta_\alpha + i\eta_\alpha$, $\alpha = 1, \cdots, n$. We have

$$dx_2 \cdots dx_2 n = \left|\frac{\partial(x_2 \cdots x_{2n})}{\partial(\eta_1, \zeta_2, \eta_2, \cdots, \zeta_n, \eta_n)}\right| \sigma(d\zeta).$$

By Definition 1, $\left|\frac{\partial(x_2\cdots x_{2n})}{\partial(\eta_1,\zeta_2,\eta_2,\cdots,\zeta_n,\eta_n)}\right| = \frac{\operatorname{Im}F}{\partial\eta_1}\neq 0$ and $\sigma(d\zeta) = \frac{1}{\left|\frac{\partial \operatorname{Im}F}{\partial\eta_1}\right|}\,|\,dx_2\cdots dx_{2n}\triangleq M\,dx_2\cdots dx_{2n}$. Since

$$|F_j(\zeta,z)\sim|\operatorname{Re}F_j(\zeta,z)|+|\operatorname{Im}F_j(\zeta,z)|, \tag{16}$$

by (10) and (11), we have

$$|\beta_\varepsilon(z)| = \left|\frac{(n-1)!}{(2\pi i)^2}\int_\varepsilon \frac{D'_{j_1\cdots j_n}}{\prod_{j=1}^n F_{jk}(\zeta,z)}d\zeta\right|.$$

By the equivaclent integral representation of Weil integral representation, we have

$$|\beta_\varepsilon(z)| \leqq \left|\frac{(n-1)!}{(2\pi i)^2}\int_{\substack{\varepsilon\times(\lambda_1+\cdots+\lambda_n)=1\\ \varepsilon>\lambda_j\geq 0}} \frac{dx_2\cdots dx_{2n}}{[(|\rho_0(z)|+x_2^{2+}\cdots+x_{2n}^2)+x_2^2]^{\frac{n}{2}}}\right.$$

$$\left.\sum_{h=1}^n(-1)^{h-1}\lambda_h d\lambda_{[h]}\right|. \tag{17}$$

Since N is finite, there exists $|\rho_0(z)| = \min\{|\rho_j(z)|, j=1,\cdots,N\}$ and

$$\int_{\substack{\lambda_1+\cdots+\lambda_n)=1\\ \varepsilon>\lambda_j\geq 0}}\sum_{h=1}^n(-1)^{h-1}\lambda_h d\lambda_{[h]}\,| = \frac{1}{(n-1)!}. \tag{18}$$

Let

$$\begin{aligned}
&x_2 = \gamma\cos\psi_1,\psi_1 = \psi,\\
&x_3 = \gamma\sin\psi_1\cos\psi_1,\cdots,x_{2n} = \gamma\sin\psi_1\sin\psi_2\cdots\gamma\sin\gamma_{2n-3}\sin\psi_{2n-2},\\
&0<\gamma\leq\gamma_0, 0\leq\psi_1=\psi\leq\pi,\cdots,0\leq\psi_{2n-2}\leq 2\pi,\\
&J = \gamma^{2n-2}\sin^{2n-3}\psi_1\sin^{2n-4}\psi_2\cdots\sin\psi_{2n-3}.
\end{aligned}$$

Thus we have

$$|\beta_\varepsilon(z)| \leqq \frac{M}{(2ni)^2}\int_{0+|\rho_0(x)|}^{\gamma_0^2+|\rho_0(x)|}\frac{d\mu}{\mu} = M_0\ln\frac{\gamma_0^2+|\rho_0(z)|}{|\rho_0(z)|},\ \gamma_0<\varepsilon. \tag{19}$$

Especially when $|\rho_0(z)| = O(\gamma_0^2)$, $\gamma_0\to 0$ and $\varepsilon^+\to 0$, the right hand side of (19) is convergent. Let

$$\beta = \lim_{\gamma_0\to 0}\ln\frac{\gamma_0^2+|\rho_0(z)|}{|\rho_0(z)|}.$$

If $|\rho_0(z)| = \underline{O}(\gamma_0^2)$, then $\beta = C$ (constant); If $|\rho_0(z)| = \bar{o}(\gamma_0^2)$, $|\rho_0(z)|\simeq c\gamma_0^v$ and $0<v<2$, then $\beta = -\infty$. If $|\rho_0(z)|\simeq c\gamma_0^v$, $v>2$ and $|\rho_0(z)|\simeq c\gamma_0^v\ln\gamma_0$, then $\beta = 0$. But by conditions (2) - (6), there exists $\beta = \beta(z)$.

Let us return to the proof Lemma 2. Let

$$\omega_\varepsilon = \Omega\cap\{\zeta|\prod_{j=1}^n|p_j(\zeta,z)\cdot(\zeta,z)|\leq\varepsilon\} \tag{20}$$

and

$$\Omega - \omega_\varepsilon = \Omega \cap \{ \zeta \mid \prod_{j=1}^{n} | p_j(\zeta, z) \cdot (\zeta, z) | \geq \varepsilon \}. \tag{21}$$

We have

$$\int_\Omega f(\zeta) k(\zeta, z) = \int_\Omega [f(\zeta) - f(z)] k(\zeta, z) + f(z) \int_\Omega k(\zeta, z).$$

Let

$$\int_{\Omega - \omega_\varepsilon} [f(\zeta) - f(z)] k(\zeta, z) = I_1, \quad f(z) \int_{\Omega - \omega_\varepsilon} k(\zeta, z) = I_2.$$

And

$$I_1 = [\int_{\omega_{\varepsilon_0} - \omega_\varepsilon} + \int_{\Omega - \omega_{\varepsilon_0}}][f(\zeta) - f(z)] k(\zeta, z) = I_1^1 + I_1^2.$$

Now we estimate I_1^1. By (20)

$$(\omega_{\varepsilon_0} - \omega_\varepsilon) \subset U(z, \delta) \cap \partial \triangle \quad (\varepsilon_0 < \delta).$$

Similar to the estimation of (10), we have by following [6], that

$$\left| \int_{\omega_{\varepsilon_0} - \omega_\varepsilon} k(\zeta, z) \right| \leq M \ln \frac{\gamma_0^2 + |\rho_0(z)|}{\gamma_1^2(\varepsilon) + |\rho_0(z)|}, \quad M > 0.$$

Since $f(\zeta) \in C_0^\infty(\Omega)_0$ Thus, $f(\zeta) \in H(\alpha, \Omega)$, $|f(\zeta) - f(z)| \leq M_1 \gamma_0^\alpha$ and

$$| I_1^1 | \leq M_2 \gamma_0^\alpha \ln \frac{\gamma_2^2 + |\rho_0(z)|}{\gamma_1^2(\varepsilon) + |\rho_0(z)|}, \quad M_2 = M M_1 > 0.$$

where $(\omega_{\varepsilon_0} - \omega_\varepsilon) \subset U(z, \delta) \cap \partial \triangle$, and $x_1 = 0$, $\gamma_1^2(\varepsilon) \leq x_2^2 + \cdots + x_{2n}^2 \leq \gamma_2$. For any $\varepsilon > 0$, we can appropriately choose $\gamma_1^2(\varepsilon)$, for example $\varepsilon \geq \gamma_3 |\rho_0(z)|$, where γ_3 is a sufficiently large constant, we also can choose $\gamma_1^2(\varepsilon) = \gamma_4 \varepsilon$. Now for fixed $\varepsilon (0 < \varepsilon \leq \gamma_5)$ where γ_5 is independent of ζ and z, we have

$$| I_1^1 | \leq A_0 \gamma_0^\alpha M_2 \ln \frac{M_3}{\varepsilon} \leq M, \quad M > 0.$$

Namely, the integral I_1^1 exists, and the integral I_1^2 is also convergent. By Theorem 1, the integral I_2 is convergent.

So the principal value (8) exists.

Smilarly, when $z \in \triangle^+$, we have

$$k^+ f(z) = k_b f(z) + \beta_1(z) f(z). \tag{22}$$

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一类解析多面体的奇异积分

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摘 要

本文利用 R. Harvey **和** J. Porking **的方法首先定义广义式的** Cauchy **主值,利用同伦公式,借助积分变换技巧研究** Weil **型积分的边界性质,得到** Plemelj **公式. 它有别于通常研究边界性质的方法. 本文引入细复广义权和** Choquet **型复广义权的概念,讨论了某些与复广义权相关的函数的拟连续性与细拟处处连续的关系.**