Journal of Mathematical Research & Exposition
Vol. 18, No. 3, 373 - 383, August 1998

A Priori Estimates for a Quasilinear Elliptic PDE *

Yang Zuodong
(Dept. of Math., Henan Normal University, Xinxiang 453002)

Abstract In the recent past many results have been established on positive solutions to boundary value problems of the form

$$-\operatorname{div}(|Du(x)|^{p-2}Du(x)) = \lambda f(u(x))\} \text{ in}\} \ \Omega,$$
$$u(x) = 0 \text{ on } \partial\Omega,$$

where $\lambda > 0$, Ω is a bounded smooth domain and $f(s) \geq 0$ for $s \geq 0$. In this paper we study a priori estimates of positive radial solutions of such problems when $N > p > 1$, $\Omega = B_1 = \{x \in \mathbf{R}^N; |x| < 1\}$ and $f \in C^1(0, \infty) \cap C^0([0, \infty))$, $f(0) = 0$.

Keywords estimates, elliptic, queasilinear.

Classification AMS(1991) 35J65, 35B25/ CCL O175.25

1. Introduction

In this paper, we consider the set of positive radial solution to the following quasilinear elliptic PDE

$$\operatorname{div}(|Du|^{p-2}Du) + \lambda f(u) = 0 \text{ in } \Omega, \tag{1.1}$$
$$u = 0 \text{ on } \partial\Omega, \tag{1.2}$$

where Ω denotes the unit ball in $\mathbf{R}^N (N \geq p)$, and $\lambda > 0$. Here $f: [0, \infty) \to \mathbf{R}$ satisfies

$$f(0) = 0, \ f'(u) \geq 0, \lim_{u \to \infty} \frac{f(u)}{u^q} = L_0 > 0, \tag{1.3}$$

$$\lim_{u \to 0} \frac{f(u)}{u^q} = L_0^* \tag{1.4}$$

for some q with $p - 1 < q < ((p-1)N + p)/(N - p)$. For $N = p$ we have $p - 1 < q < \infty$.

Problems (1.1) - (1.2) arises from many branches of mathematics and applied mathematics. The existence and uniqueness of the positive solutions of (1.1) - (1.2) have been studied by many authors. See, for example, [1 - 12] and the references therein.

* Received November 27, 1995. Supported by the Foundations of Henan Education Committee, and the Committeeon Science and Technology of Henan Province.

A natural question which then arises is to determine how λ and $d = \max_\Omega u$ are related. When $p = 2$, the related results have been obtained by [13]. In this paper, we will prove the following theorem.

Theorem 1 If u is a solution of (1.1), (1.2) with $d = \max_\Omega u$, and f satisfies (1.3) and (1.4), then for small λ we have

$$\left(\frac{p}{p-1}\right)^{p-1}(N-1) \leq \frac{\lambda f(d)}{d^{p-1}} \leq C,$$

where C is a constant that is independent of λ and d.

Assuming now that (1.1), (1.2) has a positive radial solution, let us denote it as u_λ. Letting $d_\lambda = \max_\Omega u_\lambda$, we then define $v_\lambda = u_\lambda / d_\lambda$. Then v_λ satisfies

$$\operatorname{div}(|Dv|^{p-2}Dv) + \frac{\lambda}{d_\lambda^{p-1}} f(d_\lambda v) = 0 \text{ in } \Omega, \tag{1.5}$$

$$v = 0 \text{ on } \partial\Omega, \tag{1.6}$$

$$0 < v \leq 1 \text{ in } \Omega \tag{1.7}$$

with the aid of theorem 1, we will also prove the following theorem.

Theorem 2 *If u_λ satisfies (1.1), (1.2) and f satisfies (1.3) and (1.4) then (for some subsequence) $\lim_{\lambda\to 0} v_\lambda = v$ and v is a positive solution of*

$$\operatorname{div}(|Dv|^{p-2}Dv) + L_1 v^q = 0 \text{ in } \Omega,$$

$$v = 0 \text{ on } \partial\Omega,$$

where $L_1 = \lim_{\lambda\to 0} \lambda f(d_\lambda)/d_\lambda$.

Theorem 3 Assume *that f satisfies* (1.3) and (1.4). Let $\lambda_n \to 0$ and u_n be positive radial solutions of (1.1) - (1.2) for $\lambda = \lambda_n$ such that $d = \|u_n\|_\infty \to \infty$ as $n \to \infty$. Then $(L_0 \lambda_n)^{1/(q-p+1)} u_n \to \omega_q$ in $C^1(\Omega)$ *as* $n \to \infty$, where ω_q is a positive radial solution of the problem

$$-\operatorname{div}(|D\omega|^{p-2}D\omega) = \omega^q \text{ in } \Omega, \omega = 0 \text{ on } \partial\Omega.$$

2. Preliminaries

We consider positive solution of (1.1), (1.2) are radial solutions, thus, $u = u(r, \lambda)$ satisfies

$$(\phi_p(u'))' + \frac{N-1}{r}\phi_p(u') + \lambda f(u) = 0, \tag{2.1}$$

$$u(0) = d,\ u'(0) = 0,\ u(1) = 0,\ u'(r) < 0 \text{ for } 0 < r < 1. \tag{2.2}$$

Multiplying (2.1) by u' and integrating on (0,1) gives

$$(p-1)/p\,|u'(1)|^p + \int_0^1 (N-1)/r\,|u'|^p dr = \lambda F(d), \tag{2.3}$$

where $F(u) = \int_0^u f(s)\,ds$.

Another useful identity is obtained by multiplying (2.1) by r^{N-1} and integrating on $(0, r)$. This gives

$$-r^{N-1}\phi_p(u') = \lambda \int_0^r s^{N-1} f(u(s))\,ds. \qquad (2.4)$$

Now using the fact that f is increasing, u is decreasing and $u \geq 0$, we obtain

$$-u' \geq \left(\frac{\lambda f(u)\, r}{N}\right)^{1/(p-1)}. \qquad (2.5)$$

3. Proof of Theorem

We first recall a Pohozaev identity which was obtained by Ni and Serrin[8].

Lemma 3.1 *Let $u(r)$ be a solution of (2.1) in $(r_1, r_2) \subset (0, \infty)$ and let a be an arbitrary constant. Then, for each $r \in (r_1, r_2)$ we have*

$$\frac{d}{dr}\left[r^n\left\{(1-1/p)\,|u'|^p + F(u) + \frac{a}{r}uu'|u'|^{p-2}\right\}\right] = r^{n-1}\left[nF(u) - auf(u) + (a+1) - n/p)\,|u'|^p\right].$$

Proof of Theorem 1 The left hand side of the inequality mentioned in theorem 1 above is now established with identity (2.3). First, using Holder's inequality we have

$$d = u(0) - u(1) = \int_0^1 -u'ds = \int_0^1 \frac{-u'}{\sqrt[p]{r}}\sqrt[p]{r}\,dr \leq \left(\int_0^1 (-u')^p/\,r\,dr\right)^{1/p}\left(\frac{p-1}{p}\right)^{(p-1)/p}.$$

Using (2.3) we now obtain

$$d^p \leq \left(\frac{p-1}{p}\right)^{p-1}\int_0^1 \frac{(-u')^p}{r}dr \leq \left(\frac{p-1}{p}\right)^{p-1}\frac{\lambda F(d)}{(N-1)}.$$

Finally, since $f' \geq 0$ we have

$$F(d) = \int_0^d f(s)\,ds = df(d) - \int_0^d sf'(s)\,ds \leq df(d).$$

Therefore,

$$((p-1)/p)^{p-1}(N-1) \leq \frac{\lambda f(d)}{d^{p-1}}. \qquad (3.1)$$

Thus, the left hand side of the inequality in theorem is eastablished.

To obtain the other half of theorem 1, we begin with inequality (2.5)

$$-u' \geq \left(\frac{\lambda f(u)\, r}{N}\right)^{1/(p-1)}.$$

Next, using (1.3) and (1.4) we see that there exists a $C > 0$ such that

$$-u' \geq \left(\frac{\lambda f(u)\, r}{N}\right)^{1/(p-1)} \geq \left(\frac{C\lambda r u^q}{N}\right)^{1/(p-1)}.$$

Dividing by $u^{q/(p-1)}$ and integrating this inequality we obtain

$$\frac{1}{u^{(q-p+1)/(p-1)}} - \frac{1}{d^{(q-p+1)/(p-1)}} \geq C\lambda^{1/(p-1)} r^{p/(p-1)}.$$

Therefore,

$$\left(\frac{u}{d}\right)^{(q-p+1)/(p-1)} \leq \frac{1}{1 + C\lambda^{1/(p-1)} d^{(q-p+1)/(p-1)} r^{p/(p-1)}}. \tag{3.2}$$

We will now estimate $|u'(1)/d|$ from above and below. This will in turn lead to an upper bound for $\lambda d^{q-p+1} \sim \lambda f(d)/d^{p-1}$ for small λ (and thus large d by (3.1)).

First, we estimate $|u'(1)/d|$ from above. From (2.4) and (1.3) and (1.4) we have

$$|u'(1)|^{p-1} = \lambda \int_0^1 s^{N-1} f(u)\,ds \leq C\lambda \int_0^1 s^{N-1} u^q ds.$$

Using (3.2), we obtain

$$\left|\frac{u'(1)}{d}\right|^{p-1} \leq C\lambda d^{q-p+1} \int_0^1 s^{N-1} (u/d)^q dr$$
$$\leq C\lambda d^{q-p+1} \int_0^1 \frac{r^{N-1}\,dr}{1 + C(\lambda d^{q-p+1})^{q/(q-p+1)} r^{pq/(q-p+1)}}.$$

Letting

$$s = (\lambda d^{q-p+1})^{1/p} r,$$
$$ds = (\lambda d^{q-p+1})^{1/p} dr,$$

we obtain

$$\left|\frac{u'(1)}{d}\right|^{p-1} \leq \frac{1}{C(\lambda d^{q-p+1})^{(N-p)/p}} \int_0^{(\lambda d^{q-p+1})^{1/p}} \frac{s^{N-1}\,ds}{1 + Cs^{pq/(q-p+1)}}. \tag{3.3}$$

Now we estimate $|u'(1)/d|$ from below.

We reture to (2.1), (2.2)

$$(\phi_p(u'))' + \frac{N-1}{r}\phi_p(u') + \lambda f(u) = 0, \quad 0 < r < 1,$$

$$u(0) = d > 0, \ u'(0) = 0, \ u(1) = 0, \ u'(r) < 0.$$

We define

$$E(r) = (p-1)/p\,|u'|^p(r) + \lambda F(u(r))$$

and

$$H(r) = rE(r) + (N-p)/p\,uu'(r)\,|u'|^{p-2}.$$

By Lemma 3.1, we obtain on (r_0, r_1) after a lengthy computation

$$r_1^{N-1} H(r_1) - r_0^{N-1} H(r_0) = \int_0^1 \lambda r^{N-1} [NF(u) - (N-p)/p\,uf(u)]\,dr. \tag{3.4}$$

Let t_0 be such that $d \geq u(r) \geq kd$ for all $0 \leq r \leq t_0$ and $u(t_0) = kd$ for $(\frac{N-p}{Np})^{1/(q+1)} < k < 1$. Then from (2.4) we have

$$(-u')^{p-1} = \frac{\lambda}{r^{N-1}} \int_0^r s^{N-1} f(u)\, ds$$

and hence

$$\lambda f(kd)\, r \leq N(-u')^{p-1} \leq \lambda f(d)\, r$$

on $[0, t_0]$. Integrating on $[0, t_0]$ gives

$$C_1 \left(\frac{d^{p-1}}{\lambda f(d)}\right)^{1/p} \leq t_0 \leq C_1 \left(\frac{d^{p-1}}{\lambda f(kd)}\right)^{1/p}, \tag{3.5}$$

where $C_1 = [(p/(p-1))^{p-1}(1-k)^{p-1} N]^{1/p} > 0$. Substituting $r_0 = 0$ and $r_1 = t_0$ in (3.4) and using (3.5) gives

$$\begin{aligned} t_0^{N-1} H(t_0) &\geq \int_0^{t_0} \lambda r^{N-1} (NF(u) - (N-p)/p\, uf(u))\, dr \\ &\geq \int_0^{t_0} \lambda r^{N-1} (NF(kd) - (N-p)/p\, df(d))\, dr \\ &\geq \lambda [NF(kd) - (N-p)/p\, df(d)] \frac{t_0^N}{N} \\ &\geq \frac{C_1^N}{N} [NF(kd) - (N-p)/p\, df(d)] \left(\frac{d^{p-1}}{f(d)}\right)^{N/p} \frac{1}{\lambda^{N/p}}. \end{aligned} \tag{3.6}$$

Since f satisfies (1.3) and (1.4), we then have that there exists $A \geq 0$ such that

$$NF(u) - (N-p)/p\, uf(u) \geq -A$$

for all u. Thus, we have by (1.3), (1.4), (3.4) and (3.6)

$$\begin{aligned} (p-1)/p\, |u'|^p(1) = H(1) &\geq \frac{C_1^N}{N} \lambda^{(p-N)/p} [NF(kd) - (N-p)/p\, f(d)\, d] \left(\frac{d^{p-1}}{f(d)}\right)^{N/p} \\ &\quad - A\lambda(1 - t_0^N)/N \geq c\lambda^{(p-N)/p} d^{[N(p-1)+p-q(N-p)]/p} - A\lambda \\ &\geq c\lambda^{(p-N)/p} d^{[N(p-1)+p-q(N-p)]/p}. \end{aligned}$$

Therefore,

$$\frac{|u'|^p(1)}{d^p} \geq \frac{c}{(\lambda d^{q-p+1})^{(N-p)/p}}. \tag{3.7}$$

Hence, combining (3.3) and (3.7) gives

$$\begin{aligned} \frac{C}{(\lambda d^{q-p+1})^{(N-p)/p}} &\leq |u'(1)/d|^p \\ &\leq \frac{C}{(\lambda d^{q-p+1})^{(N-p)/(p-1)}} \left(\int_0^{(\lambda d^{q-p+1})^{1/p}} \frac{s^{N-1}\, ds}{1 + Cs^{pq/(q-p+1)}}\right)^{p/(p-1)}. \end{aligned} \tag{3.8}$$

Now we let $T = (\lambda d^{q-p+1})^{1/p}$, and by rewriting (3.8) we see that we have

$$CT^{(N-p)/(p-1)} \leq \left(\int_0^T \frac{s^{N-1}\, ds}{1 + Cs^{pq/(q-p+1)}}\right)^{p/(p-1)}. \tag{3.9}$$

We want to show now that T is bounded as $\lambda \to 0$. We need to consider separately the cases $N > p$ and $N = p$.

Case I $N > p$

Suppose now that $T \to 0$ as $\lambda \to 0$. We will show that is impossible. First, if the term on the right is bounded then we are done (this is the case if $q < (p-1)N/(N-p)$).

Assuming, therefore, that is term goes to infinity as $\lambda \to 0$ we divide both sides by $T^{(N-p)}$ and get

$$0 < C^{p-1} < \left(\frac{\int_0^T s^{N-1}\,ds/(1+Cs^{pq/(q-p+1)})}{T^{(N-p)/p}}\right)^p.$$

We will now show that if $T \to \infty$ as $\lambda \to 0$ then the right hand side of the above goes to zero and thus contradicts the above inequality. Applying L'Hopital's rule we obtain

$$\begin{aligned}
0 < C^{p-1} &\le \left(\lim_{T\to\infty}\frac{\int_0^T s^{N-1}\,ds/(1+Cs^{pq/(q-p+1)})}{T^{(N-p)/p}}\right)^p \\
&= \left(\lim_{T\to\infty}\frac{T^{N-1}/(1+CT^{pq/(q-p+1)})}{(N-p)/pT^{(N-2p)/p}}\right)^p \\
&= \left(\frac{p}{N-p}\right)^p\left(\lim_{T\to\infty}\frac{T^{[(p-1)N+p]/p}}{1+CT^{pq/(q-p+1)}}\right)^p \\
&= \left(\frac{p}{N-p}\right)^p\left(\lim_{T\to\infty}\frac{1}{T^{-[(p-1)N+p]/p}+CT^{[q(N-p)-(p-1)N-p]/p(q-p+1)}}\right)^p.
\end{aligned}$$

As $T \to \infty$ this goes to zero for $p-1 < q < [(p-1)N+p]/(N-p)$ and $N > p$. Hence we obtain the desired contradiction.

Case II $N = P$

For $N = p$, we can only conclude from (3.3) ad (3.7) (since $A\lambda/d^p \to 0$ as $\lambda \to 0$) that

$$\begin{aligned}
0 < c_1 < \left|\frac{u\lambda(1)}{d}\right| &\le \left(C\int_0^T \frac{s^{N-1}\,ds}{1+Cs^{pq/(q-p+1)}}\right)^{1/(p-1)} \\
&\le \left(C\int_0^\infty \frac{s^{N-1}\,ds}{1+Cs^{pq/(q-p+1)}}\right)^{1/(p-1)} \le c_2 < \infty,
\end{aligned}$$

where c_1, c_2 are independent of λ. We now define

$$v\lambda = \frac{u\lambda}{d\lambda}.$$

Assuming that $T \to \infty$ as $\lambda \to 0$ we conclude from (3.2) that

$$v\lambda \to 0 \tag{3.10}$$

uniformly say for $1/2 \le r \le 1$.

Claim $v\lambda$, $v\lambda'$, $(\phi_p(v\lambda'))'$ are uniformly bounded on $[1/2, 1]$.

Once the claim is proven, we can then conclude that there is a subsequence with $v\lambda \to v$ and $v\lambda' \to v'$ uniformly on $[1/2, 1]$. From (3.10), we have that $v \equiv 0$ on $[1/2, 1]$. On the other hand, we have that

$$0 < c_1 \le |v\lambda'(1)| \le c_2.$$

Therefore, $v'(1) = \lim_{\lambda\to 0} v_\lambda'(1) \le -c_1 < 0$. Thus, $v > 0$ on $(1-\varepsilon, 1)$. This contradicts that $v \equiv 0$ on $[1/2, 1]$. Thus, we only need to prove the claim.

Proof of Claim Recalling (2.4) and (3.2) gives

$$r^{N-1}(-v')^{p-1} = \frac{\lambda}{d^{p-1}}\int_0^r r^{N-1} f(u)\,dr \le c\lambda d^{q-p+1} r^{N-p}\int_0^r s^{p-1} v^q ds$$

$$\le \lambda d^{q-p+1} r^{N-p}\int_0^r \frac{s^{p-1}\,ds}{1 + C(\lambda d^{q-p+1})^{q/(q-p+1)} s^{pq/(q-p+1)}}.$$

Letting

$$t = (\lambda d^{q-p+1})^{1/p} s,$$

$$dt = (\lambda d^{q-p+1})^{1/p} ds,$$

gives

$$(-rv')^{p-1} \le \lambda d^{q-p+1}\int_0^{(\lambda d^{q-p+1})^{1/p} r} \frac{t^{p-1}\,dt}{1 + Ct^{pq/(q-p+1)}} \le \int_0^\infty \frac{t^{p-1}\,dt}{1 + Ct^{pq/(q-p+1)}} = B < \infty.$$

Thus, for $r \in [1/2, 1]$ we have

$$|v'| \le B,$$

where B is independent of λ. From the differential equation we have

$$|(\phi_p(v'))'| = \frac{N-1}{r}|v'|^{p-1} + \frac{\lambda}{d^{p-1}} f(vd).$$

Using (3.2) again gives

$$|(\phi_p(v'))'| \le B^{p-1}(N-1)/r + C\lambda v^q d^{q-p+1}$$

$$\le B^{p-1}(N-1)/r + \frac{\lambda d^{q-p+1}}{C(\lambda d^{q-p+1})^{q/(q-p+1)} r^{pq/(q-p+1)}}$$

$$\le B^{p-1}(N-1)/r + \frac{1}{(\lambda d^{q-p+1})^{(p-1)/(q-p+1)} r^{pq/(q-p+1)}} \le C$$

on $[1/2, 1]$ because by assumption $\lambda d^{q-p+1} \to \infty$ as $\lambda \to 0$. Thus

$$v, |v'|, |(\phi_p(v'))'| \le C$$

and this completes the proof of the claim.

Proof of Theorem 2 From (2.4) we have that

$$r^{N-1}(-v')^{p-1} = \frac{\lambda}{d^{p-1}}\int_0^r s^{N-1} f(v_\lambda d)\,ds.$$

From theoerm 1, we have that $\lambda f(d)/d^{p-1} \le C$. Thus, since $f'(u) \ge 0$ we have that

$$r^{N-1}(-v')^{p-1} \le C\frac{r^N}{N}.$$

Hence ,

$$\frac{| v' |^{p-1}}{r} \leq C.$$

Substituting back into (2.1) we further obtain

$$| (\phi_p(v\lambda))' | \leq \frac{N-1}{r} | v\lambda |^{p-1} + \frac{\lambda f(d)}{d^{p-1}} \leq C.$$

so combining the above and recaling that $0 \leq v\lambda \leq 1$ we obtain that

$$v\lambda, | v\lambda | , | (\phi_p(v\lambda))' | \leq C.$$

Thus , by the Arzela - Ascoli theorem , there is a subsequence of the λ (still denoted λ) such that

$$\phi_p(v\lambda) \to \omega \Rightarrow v\lambda \to \phi_p^{-1}(\omega) = v' \text{uniformly} ,$$

$$v\lambda = \int v\lambda(s)\, ds \to \int \phi_p^{-1}(\omega)\, ds = v \text{ uniformly}.$$

For some further subsequence (again denoted λ) , we have

$$\lim_{\lambda \to 0} \frac{\lambda f(d)}{d^{p-1}} = L_1 < \infty.$$

Since $v\lambda \to v$ uniformly on $[0,1]$, we also have that $v(0) = 1$ and $v(1) = 0$. Now , since $L_1 < \infty$ we have by (3.8) that $-v'(1) = \lim_{\lambda \to 0} v\lambda(1) = c > 0$. Thus , for all $\phi > 0$ there exists $\delta > 0$ such that $v\lambda \geq \varepsilon$ for $0 \leq r \leq 1 - \delta$. Returning to

$$(-v\lambda)^{p-1} = \frac{\lambda}{r^{N-1} d^{p-1}} \int r^{N-1} f(v\lambda d)\, dr ,$$

for $0 < r \leq 1 - \delta < 1$, this converges to

$$(-v)^{p-1} = \frac{L_1}{r^{N-1}} \int r^{N-1} v^q dr.$$

Thus , v is a solution of

$$(\phi_p(v))' + \frac{N-1}{r} \phi_p(v) + L_1 v^q = 0$$

on $0 < r \leq 1 - \delta$. This holds for all $\delta > 0$. Now as $\phi \to 0$ also $\delta \to 0$, thus v is a solution on $0 < r < 1$. Further , $v > 0$ on $[0,1]$. This completes the proof of Theorem 2.

Proof of Theorem 3 Let $u_n = \| u_n \|_\infty u_n$. Then , $v_n(r)$ satisfies $\| v_n \|_\infty = 1$ and

$$-(r^{N-1} | v_n' |^{p-2} v_n')' = \lambda_n r^{N-1} (\| u_n \|_\infty)^{q-p+1} [f(\| u_n \|_\infty) / \| u_n \|_\infty^q] \text{ in } (0,1) ,$$

$$v_n'(0) = 0 , \ v_n(1) = 0.$$

We shall show that the limit of $\{\lambda_n \|u_n\|_\infty^{q-p+1}\}$ is a non-zero number as $n\to\infty$. In fact, we shall see that $\lambda_n \|u_n\|_n^{q-p+1} \not\to 0$ as $n\to\infty$. On the contary, if $\lambda_n \|u_n\|_\infty^{q-p+1}\to 0$ as $n\to\infty$, then let $\lambda_n \|u_n\|_\infty^{q-p+1} = T_n$ and $T_n\to 0$ as $n\to\infty$. Thus,

$$-\operatorname{div}(|Dv_n|^{p-2}Dv_n) = T_n[f(\|u_n\|_\infty v_n)/\|u_n\|_\infty^q].$$

This and the regularity of $-\operatorname{div}(|D|^{p-2}D)$ implies that $v_n\to 0$ as $n\to\infty$, since $\{f(\|u_n\|_\infty v_n)/\|u_n\|_\infty^q\}$ is uniformly bounded. This contradicts the fact that $\|v_n\| = \infty = 1$ for all n. By proof of theorm 1, T_n is a bounded. Let $z_n = (L_0 T_n)^{1/(q-p+1)} v_n$ and $T = \lim_{n\to\infty} T_n$. Then, the arguments above imply that $z_n\to z$ in $C^1(\Omega)$ and $z = (L_0 T)^{1/(q-p+1)} v$. Here v satisfies

$$-(r^{N-1}|v'|^{p-2}v')' = L_0 r^{N-1} v^q \; in \; (0,1),$$

$$v'(0) = 0, \; v(1) = 0.$$

Hence, z satisfies the problem

$$-(r^{N-1}|z'|^{p-2}z')' = r^{N-1}z^q \text{ in } (0,1)$$

and

$$z'(0) = 0, \quad z(1) = 0.$$

This implies that

$$(L_0\lambda_n)^{1/(q-p+1)} u_n \to z \text{ in } \Omega.$$

This completes the proof of this theorem.

4. The N = 1 Case

For $N=1$, equations (1.1), (1.2) become

$$(\phi_p(u'))' + \lambda f(u) = 0 \text{ in } 0 < r < 1,$$

$$u(0) = 0, \; u(1) = 0, \; u'(0) = A.$$

Note that u must be symmetric about $r = 1/2$. We therefore denote $u(1/2) = d = \max_\Omega u$. Multiplying by u' and integrating on $(0, r)$ gives

$$\frac{p-1}{p}|u'|^p(r) + \lambda F(u) = \frac{p-1}{p}A^p,$$

where $F(u) = \int_0^u f(s)\,ds$. Thus, at $r = 1/2$ we obtain

$$p/(p-1)\lambda F(d) = A^p.$$

Thus,

$$|u'|^p = p/(p-1)\lambda[F(d) - F(u)].$$

Therefore, integrating on $(0,1/2)$, we have

$$\int_0^{1/2}\frac{u'dr}{\sqrt[p]{F(d)-F(u)}} = 1/2\sqrt[p]{\frac{p\lambda}{p-1}}.$$

Letting $s = u(r)$, then $ds = u'(r)\,dr$, gives

$$\int_0^d \frac{ds}{\sqrt[p]{F(d) - F(s)}} = 1/2\sqrt[p]{\frac{p\lambda}{p-1}}.$$

Thus,

$$\int_0^d \sqrt[p]{\frac{f(d)\,d}{F(d) - F(s)}}\,ds = 1/2\sqrt[p]{\frac{p\lambda f(d)\,d}{p-1}}.$$

Letting $t = s/d$ gives $dt = ds/d$ and, hence

$$\int_0^1 \sqrt[p]{\frac{f(d)\,d}{F(d) - F(td)}}\,dt = 1/2\sqrt[p]{\frac{p\lambda f(d)}{(p-1)\,d^{p-1}}}.$$

Thus, to show that $\lambda f(d)/d^{p-1}$ is bounded, we only need to consider the left hand side of the above equation. Using (1.3) it is straightforward to see that

$$\lim_{d\to\infty}\frac{f(d)\,d}{F(d) - F(td)} = \frac{q+1}{1 - t^{q+1}}.$$

Thus, we would like to such that

$$\lim_{d\to\infty}\int_0^1 \sqrt[p]{\frac{f(d)\,d}{F(d) - F(td)}}\,dt = \sqrt[p]{q+1}\int_0^1 \frac{dt}{\sqrt[p]{1 - t^{q+1}}}.$$

In order to do this, we will break the integral into two pieces and show that each is finite. For $0 \le t \le 1/2$ we have

$$\limsup_{d\to\infty}\int_0^{1/2} \sqrt[p]{\frac{f(d)\,d}{F(d) - F(td)}}\,dt \le \limsup_{d\to\infty}\sqrt[p]{\frac{f(d)\,d}{2^p(F(d) - F(1/2\,d))}}$$

$$= \sqrt[p]{\frac{2^{q+1}(q+1)}{2^{p+q+1} - 2^p}}.$$

For $1/2 \le t \le 1$ we have $F(d) - F(td) = f(c)(1-t)\,d$, where $td \le c \le d$. Since f is increasing we have $f(c) \ge f(td)$. Thus,

$$\limsup_{d\to\infty}\int_{1/2}^1 \sqrt[p]{\frac{f(d)\,d}{F(d) - F(td)}}\,dt \le \limsup_{d\to\infty}\int_{1/2}^1 \sqrt[p]{\frac{f(d)\,d}{f(td)(1-t)\,d}}$$

$$\le \limsup_{d\to\infty}\sqrt[p]{\frac{f(d)}{f(d/2)}}\int_{1/2}^1 \frac{dt}{\sqrt[p]{1-t}} = p/(p-1)\,2^{[(q-1)p+1]}.$$

Thus, by the dominated convergence theorem we have

$$\lim_{\lambda\to 0}\sqrt[p]{\frac{p\lambda f(d)}{(p-1)\,d^{p-1}}} = \lim_{d\to\infty}\int_0^1 \sqrt[p]{\frac{f(d)\,d}{F(d) - F(td)}}\,dt = \sqrt[p]{q+1}\int_0^1 \frac{dt}{\sqrt[p]{1 - t^{q+1}}} < \infty.$$

The proof of Theorem 2 for the case $N = 1$ is similar to the proof for $N \ge p$ decribed earlier.

References

[1] M. Guedda and L. Veron, *Local and global properties of quasilinear elliptic equations*, J. Diff. Eqns., **76**(1988), 159 - 189.

[2] Z. M. Guo, *Existence and uniqueness of the positive radial solutions for a class of quasilinear elliptic equations*, Appl. Anal., **47**(1992), 173 - 190.

[3] Z. M. Guo, *Some existence and multiplicity results for a class of quasilinear elliptic equations*, Nonlinear Anal., **18**:**10**(1992), 957 - 971.

[4] Z. M. Guo, *Boundary value problems of a class of quasilinear ordinary differential equations*, Diff. Intergral Eqns., **6**:**3**(1993), 705 - 719.

[5] Z. M. Guo and J. R. L. Webb, *Uniqueness of positive solutions for quasilinear elliptic equations when a parameter is large*, Proc. Roy. Soc. Edinburgh **124**A(1994), 189 - 198.

[6] Z. M. Guo, *On the positive solutions for a class of quasilinear non - positive problems*, Chinese quarterly J. of Math., **12**(1996), 1 - 11.

[7] Z. M. Guo and Z. D. Yang, *Structure of positive solutions for quasilinear elliptic equations when a parameter is small*, Chinese Ann. Math, to appear.

[8] Z. D. Yang and Z. M. Guo, *On the structure of positive solutions for quasilinear ordinary differential equations*, Appl. Anal., **58**(1995), 31 - 51.

[9] Z. D. Yang and H. S. Yang, *Periodic boundary value problems of a class quasilinear ordinary differential equations*, Ann. of Diff. Eqs., **10**:**4**(1995), 464 - 471.

[10] W. M. Ni and J. Serrin, *Nonexistence theorems for singular solutions of quasilinear partial differential equations*, Comm. Pure Appl. Math., **39**(1986), 379 - 399.

[11] M. N. Pino, M. Elgueta and R. Manasevich, *A homotopic deformation along p of a Leray - Schauder degree result and existence for* $(|u'|^{p-2}u')' + f(t,u) = 0$, $u(0) = u(T) = 0$, $p > 1$, J. diff. Eqns., **80**(1989), 1 - 13.

[12] P. Tolksdorf, *Regularity of a more general class of quasilinear elliptic equations*, J. Diff. Eqns., **51**(1984), 126 - 150.

[13] J. A. Iaia, *A priori estimates for a semilinear elliptic PDE*, Nonlinear Anal., **24**:**7**(1995), 1039 - 1048.

一类拟线性椭圆型偏微分方程的先验界的估计

杨 作 东

(河南师范大学数学系, 新乡 453002)

摘 要

近几年对边值问题

$$-\operatorname{div}(|Du|^{p-2}Du) = \lambda f(u)\} \text{ 在 } \Omega \text{ 上}$$

$$u|_{\partial\Omega} = 0$$

正解方面已经得到了许多结果. 这里 $\lambda > 0$, Ω **是有界区域和对** $s \geq 0$, $f(s) \geq 0$. **在本文中在条件** $N \geq p > 1$, $\Omega = B_1 = \{x \in R^N, |x| < 1\}$ **和** $f \in C^1(0,\infty) \cap C^0([0,\infty))$, $f(0) = 0$, **研究了这类问题的正对称解的先验界估计**.