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On the Equivalences of Matrix Norms Related to the l_p Norms*

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Abstract This note established equivalence relations among the matrix norms related to the l_p norm and the l_p operator norm for $1\le p\le\infty$, which completes the comparison results in [1, 2].

Keywords l_p norm, l_p operator norm, mixed norm, equivalence.

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1. Introduction

For a positive number $1\le p\le\infty$ and a complex matrix $A=(a_{ij})\in C_{n\times n}$, we denote by

$$|A|_p=\left(\sum_{i,j=1}^{n}|a_{ij}|^p\right)^{\frac{1}{p}}$$

the l_p norm of the matrix A, and by $\|A\|_p=\max\{|Ax|_p: \quad x\in C^n, \quad |x|_p=1\}$ the l_p operator norm of the matrix A, where $|x|_p=\left((\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}}$ with $x=(x_1,x_2,\ldots,x_n)^T$ (T denoting the transpose). For these two norms, Tasci[2] proved the following comparison result, which is a conjecture of Goldberg and Newman[1]:

Theorem 1 *Let p satisfy $1\le p\le\infty$. Then for all $A\in C_{n\times n}$ it holds that*

$$|A|_p\le n^{\frac{1}{p}}\|A\|_p,$$

where the inequality is sharp.

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Tasci also confirmed that this theorem naturally leads to a simple comparison of the l_{pq} norm, defined as

$$\|A\|_{pq} = \left[\sum_{j=1}^{n}\left(\sum_{i=1}^{n}|a_{ij}|^{p}\right)^{\frac{q}{p}}\right]^{\frac{1}{q}},$$

and the l_q operator norm.

Theorem 2 *Let p, q satisfy $1 \le p, q \le \infty$. Then for all $A \in C_{n\times n}$ it holds that*

$$\|A\|_{pq} \le \sigma_{pq}(n)\,\|A\|_q,$$

where

$$\sigma_{pq}(n) = \begin{cases} n^{\frac{1}{q}}, & \text{for } p \ge q \\ n^{\frac{1}{p}}, & \text{for } p \le q. \end{cases}$$

The main purpose of this note is to establish the reverse relations of the inequalities in Theorem 1 and Theorem 2, so that the comparisons of l_p norms, l_p operator norms as well as the mixed norms are completed, and to generalize these results so that the relations between the l_p and l_q norms as well as between the l_p and l_q operator norms can be established.

2 The main results

The main results of this note are Theorem 3 and Theorem 4 given below. In their proofs, we will use the known inequality

$$|x|_p \le \lambda_{pq}(n)\,|x|_q, \qquad \lambda_{pq}(n) = \begin{cases} 1, & \text{for } p \ge q \\ n^{\frac{1}{p}-\frac{1}{q}}, & \text{for } p \le q \end{cases}, \qquad 1 \le p, q \le \infty$$

or

$$|x|_q \le \lambda_{qp}(n)\,|x|_p, \qquad \lambda_{qp}(n) = \begin{cases} 1, & \text{for } q \ge p \\ n^{\frac{1}{q}-\frac{1}{p}}, & \text{for } q \le p \end{cases}, \qquad 1 \le p, q \le \infty.$$

Note that $\sigma_{pq}(n) = n^{\frac{1}{q}}\lambda_{pq}(n) = n^{\frac{1}{p}}\lambda_{qp}(n) = \sigma_{qp}(n)$ hold for all $1 \le p, q \le \infty$.

Theorem 3 *Let p satisfy $1 \le p \le \infty$. Then for all $A \in C_{n\times n}$ it holds that*

$$\|A\|_p \le \beta_p(n)\,|A|_p,$$

where

$$\beta_p(n) = \begin{cases} 1, & \text{for } p \le 2 \\ n^{1-\frac{2}{p}}, & \text{for } p \ge 2. \end{cases}$$

Proof Because of $1 \le p \le \infty$, we see that there exists a unique positive number $q \ge 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Noticing that $\beta_p(n) = \lambda_{qp}(n)$ holds at this time. Considering the Hölder's inequality, we can get for any $x \in C^n$, $\quad x = \sum_{i=1}^{n} x_i e_i$ that

$$|Ax|_p = \left|\sum_{i=1}^{n} x_i A e_i\right|_p \leq \sum_{i=1}^{n} |x_i| \, |Ae_i|_p \leq \left(\sum_{i=1}^{n} |x_i|^q\right)^{\frac{1}{q}} \left(\sum_{i=1}^{n} |Ae_i|_p^p\right)^{\frac{1}{p}}$$
$$= |x|_q \left(\sum_{i,j=1}^{n} |a_{ij}|^p\right)^{\frac{1}{p}} = |A|_p |x|_q \leq \beta_p(n) |A|_p |x|_p,$$

where $e_1, e_2, \ldots, e_n$ are the standard basis of C^n. Therefore, the conclusion of Theorem 3 is true.

Theorem 3 naturally results in the following result which describes the relation between the operator norm and the mixed norm.

Theorem 4 *Let p, q satisfy $1 \leq p, q \leq \infty$. Then for all $A \in C_{n \times n}$ it holds that*

$$|A|_{pq} \geq \frac{n^{\frac{1}{p}}}{\beta_q(n)\, \sigma_{pq}(n)} \|A\|_q.$$

Proof We first note that $|x|_q \leq \lambda_{pq}(n) |x|_p$ and $|A|_q = |A|_{qq} = |(|Ae_1|_q, |Ae_2|_q, \ldots, |Ae_n|_q)|_q$ from the definitions. Since

$$|A|_{pq} = |(|Ae_1|_p, |Ae_2|_p, \ldots, |Ae_n|_p)|_q \geq \frac{1}{\lambda_{qp}(n)} |(|Ae_1|_q, |Ae_2|_q, \ldots, |Ae_n|_q)|_q$$
$$= \frac{1}{\lambda_{qp}(n)} |A|_q \geq \frac{1}{\lambda_{qp}(n)\, \beta_q(n)} \|A\|_q,$$

we immediately know that the conclusion of Theorem 4 is correct.

Based on the above theorems, we can easily get the following corollaries.

Corollary 1 *Let p satisfy $1 \leq p \leq \infty$. Then for all $A \in C_{n \times n}$ it holds that*

$$\frac{1}{\beta_p(n)} \|A\|_p \leq |A|_p \leq n^{\frac{1}{p}} \|A\|_p.$$

where the inequalities are sharp.

Corollary 2 *Let p, q satisfy $1 \leq p, q \leq \infty$. Then for all $A \in C_{n \times n}$ it holds that*

$$\frac{n^{\frac{1}{p}}}{\beta_q(n)\, \sigma_{pq}(n)} \|A\|_q \leq |A|_{pq} \leq \sigma_{pq}(n) \|A\|_q.$$

3 Further generalizations

In this section, we will set up the comparison relations among l_p and l_q norms, l_p and l_q operator norms as well as the mixed norms. For this purpose, we first prove the following theorem.

Theorem 5 *Let p, q satisfy $1 \leq p, q \leq \infty$. Then for all $A \in C_{n \times n}$ it holds that*

$$\frac{n^{\frac{1}{p}-\frac{1}{q}}}{\sigma_{pq}(n)} |A|_q \leq \|A\|_p \leq \frac{n^{\frac{1}{q}}}{\beta_p(n)\sigma_{pq}(n)} |A|_q.$$

Proof We first verify the left-hand inequality. From the proof of the Theorem in [2] we can get $|A|_q \leq n^{\frac{1}{q}} \max\limits_{1 \leq i, j \leq n} |Ae_j|_q$. Considering again $|x|_q \leq \lambda_{qp}(n)|x|_p\ (\forall x \in C^n)$ we have

$$|A|_q \leq n^{\frac{1}{q}} \lambda_{qp}(n) \max_{1 \leq i, j \leq n} |Ae_j|_p \leq n^{\frac{1}{q}-\frac{1}{p}} \sigma_{pq}(n) \max_{|x|_p = 1} |Ax|_p = n^{\frac{1}{q}-\frac{1}{p}} \sigma_{pq}(n) \|A\|_p,$$

that is, the left-hand inequality of this theorem holds.

Now, we turn to the right-hand inequality. From the proof of Theorem 4 and considering the inequality $|x|_p \leq \lambda_{pq}(n)|x|_q\ (\forall x \in C^n)$, we can get $|A|_p \leq \lambda_{pq}(n)|A|_q$. This inequality combining with Theorem 3 immediately gives the inequality what we are testing.

Corollary 3 *Let p, q satisfy $1 \leq p, q \leq \infty$. Then for all $A \in C_{n \times n}$ it holds that*

$$\frac{1}{\lambda_{qp}(n)} |A|_q \leq |A|_p \leq \lambda_{pq}(n) |A|_q.$$

Theorem 5 with either Corollary 1 or Corollary 2 can result in the conclusions in the following corollary, respectively.

Corollary 4 *Let p, q satisfy $1 \leq p, q \leq \infty$. Then for all $A \in C_{n \times n}$ it holds that*

$$\frac{n^{\frac{1}{p}-\frac{1}{q}}}{\beta_q(n)\sigma_{pq}(n)} \|A\|_q \leq \|A\|_p \leq \beta_p(n)\sigma_{pq}(n) \|A\|_q.$$

$$\frac{n^{\frac{1}{q}}}{\beta_q(n)\sigma_{pq}(n)^2} |A|_p \leq |A|_{pq} \leq n^{-\frac{1}{p}} \beta_q(n)\sigma_{pq}(n)^2 |A|_p;$$

$$\frac{n^{\frac{1}{p}-\frac{1}{q}}}{\beta_q(n)\sigma_{pq}(n)} |A|_q \leq |A|_{pq} \leq \beta_q(n)\sigma_{pq}(n) |A|_q.$$

References

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2 Tasci D. *On the conjecture by Goldberg and Newman*. Linear Algebra Appl, 1995, **215**: 275-277