Journal of Mathematical Research & Exposition
Vol. 18, No. 4, 523-528, November 1998

The Disturbance Rejection in Singular Systems *

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Abstract Using linear state feedback and linear feedforward control of the disturbances, the problem of achieving the distrubance rejection in a linear time invariant singular system is treated and a constructive solvability condition is presented.

Keywords singular system, disturbance rejection, invariant subspace, feedforward control, state feedback, subspace recursion.

Classification AMS(1991) 93C45/CCL TP273

I Introduction

In recent years, there has been a growing interest in the system-theoretic problem of singular systems (or generalized state-space systems) due to the extensive applications of singular systems in large-scale systems, singular perturbation theory, circuits, economics, control theory, and other areas

Disturbance rejection is a feedback synthesis problem concerning a system with a disturbance as an input. The purpose of disturbance rejection is to apply feedback in such a way that the disturbance has no influence on the controlled output of the closed-loop system. The development of necessary and sufficient conditions for the solvability of this problem for state-space systems by Wonham et al has profound influence on geometry control theory, particularly by means of an (A, B)-invariant subspace.

In this paper, we study disturbance rejection for the singular systems. Let $X = R^n, U = R^m, Y = R^p, D = R^l$. We consider the time-invariant singular system of the form

$$E\mathring{x}(t) = Ax(t) + Bu(t) + S_1 d(t), \tag{1.1a}$$
$$y(t) = Cx(t), \tag{1.1b}$$

where $E, A: X \to X, B: U \to X, C: X \to Y, S_1: D \to X$ are linear maps with E singular. $x(t), y(t), u(t), d(t)$ are functions of time with values in X, Y, U, D respectively, $d(t)$ represents the disturbance as an input to the system. We investigate the questions of when and how the linear map $F: X \to U, F_1: D \to X$ in the state feedback and linear feedforward control of the disturbances

* Received October 30, 1995.

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$$u(t) = -(Fx(t) + F_1 d(t)) \tag{1.2}$$

may be chosen so that the output $y(t)$ of the closed-loop system

$$E\dot{x}(t) = (A - BF)x(t) + Sd(t) \tag{1.3a}$$
$$y(t) = Cx(t) \tag{1.3b}$$

is independent of $d(t)$, and the closed-loop system enjoys uniqueness of solution, where we denote $S \triangleq S_1 - BF$. In the state space case, that is when E is a nonsingular square matrix, this problem was discussed by [10]. Without considering the linear feedback control of the disturbances, this type of problem was considered in detail in chapter 4 of [2], and a complete solution is presented. Some results on disturbance rejection in singular systems are given in [4], but there the class of feedbacks considered is different from ours as defined in (1.2). In the spirit of Wonham [2], Fletcher L R and Asaraai A A. [1] developed the necessary and sufficient codition for solvability of this similar problem, but their result is far from being explicit and the control used is the state feedback.

Note that we do not need to assume that the open-loop system enjoys uniqueness of solution, but it is essential in all approaches that F be chosen so that uniqueness of solutions to (1.3a) prevails, so we need to avoid at the outset the pathological situation that for no F does closed loop regularity hold. It is shown in [3] that this is equuivalent to the Regularisability Assumption. We will assume that for at least one complex number λ, the matrix $[A - \lambda E, B]$ has (at least) n linearly independent columns

For a given linear map M, we also denote the matrix representation M. The image set of M is denoted by $\operatorname{Im} M$, the kernel of M is denoted by $\operatorname{Ker} M$. superscipt -1 denotes inverse image of a linear map. We adopt the following key concept:

Definition 1.1 *A pair of subspaces $\mathcal{U}, \mathcal{V}$ is said to be $(\{A, E\}, B)$-invariant subspace pair contained in* Ker C, *if* (i) $A\mathcal{V} \subseteq E\mathcal{V} + \operatorname{Im}B$, $\mathcal{V} \subseteq \operatorname{Ker}C$; (ii) $E\mathcal{U} \subseteq A\mathcal{U} + \operatorname{Im}B$, $\mathcal{U} \subseteq \operatorname{Ker}C$.

We define the classes of subspaces

$$\mathbf{T}_1(\{A, E\}, B) \triangleq \{\mathcal{V} \subseteq X \mid \mathcal{V} \subseteq \operatorname{Ker}C,\ A\mathcal{V} \subseteq E\mathcal{V} + \operatorname{Im}B\}.$$
$$\mathbf{T}_2(\{A, E\}, B) \triangleq \{\mathcal{U} \subseteq X \mid \mathcal{U} \subseteq \operatorname{Ker}C,\ E\mathcal{U} \subseteq A\mathcal{U} + \operatorname{Im}B\}.$$

$\mathbf{T}_1(\{A, E\}, B)$ *and* $\mathbf{T}_2(\{A, E\}, B)$ *are both closed under addition. So they have their largest members. We symbolize the supremal of* $\mathbf{T}_1(\{A, E\}, B)$, $\mathbf{T}_2(\{A, E\}, B)$ *as* $\mathbf{T}_1^*$, T_2^* *seperately. The computation of* $\mathbf{T}_1^*$, $\mathbf{T}_2^*$ is given in the next section. For $\mathbf{T}_1^*$, $\mathbf{T}_2^*$ we define the classes of friends as follows

$$\mathbf{F}(\mathbf{T}_1^*) \triangleq \{F: X \to U \mid (A - BF)\mathbf{T}_1^* \subseteq ET_1^*\},$$
$$\mathbf{F}(\mathbf{T}_2^*) \triangleq \{F: X \to U \mid E\mathbf{T}_2^* \subseteq (A - BF)\mathbf{T}_2^*\}.$$

From Theorem 2.1 of [1], we can easily obtain the following result.

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Theorem 1.1 *If*

(*a*) $\mathbf{T}_1^* \cap \mathrm{Ker} E = 0$,

(*b*) $\dim(E\mathbf{T}_2^* \cap \mathrm{Im}\, B) \le \dim\{u \in \mathbf{T}_2^*: Au \in \mathrm{Im} B\}$, *then there exists a linear map* $F \in \mathbf{F}(\mathbf{T}_1^*) \cap (\mathbf{F}\mathbf{T}_2^*)$, *such that* $A - BF - \lambda E$ *has linearly independent column for some complex number* λ.

Proof From Theorem 2.1 of [1], for $\mathbf{T}_1^*, \mathbf{T}_2^*$ satisfying (a) (b), there exists a linear map F and a subspace W, with $\mathbf{T}_1^* \subseteq W \subseteq \mathbf{T}_1^* + \mathbf{T}_2^*$ such that

$$(A - BF)W \subseteq EW, \quad E\mathbf{T}_2^* \subseteq (A - BF)\mathbf{T}_2^*$$

and the matrix $A - BF - \lambda E$ has linearly independent columns for some complex number λ. We see that $W \subseteq \mathbf{T}_1^* + \mathbf{T}_2^* \subseteq \mathrm{Ker}\, C$, $AW \subseteq EW + \mathrm{Im} B$, so $\omega \in \mathbf{T}_1^*(\{A, E\}, B)$. Since $\mathbf{T}_1^*$ is the supremal member of $\mathbf{T}_1(\{A, E\}, B)$, we have $W = \mathbf{T}_1^*$. □

II Conditions for the solvability of the problem: geometric characterization

In this section, we will study the computation of $\mathbf{T}_1^*, \mathbf{T}_2^*$ by means of subspace recursins. Finally, geometric characterization for the solvability of the disturbance rejection problem for singular systems will be presented.

We resume our discussion by introducing a number of subspace recursions. First, We define $T^{(k)}$ by $T^{(k+1)} = A(E^{-1}(T^{(k)} + \mathrm{Im} B) \cap \mathrm{Ker}\, C)$, $T^{(0)} = 0$. It is trivial to show that $\lim_{k\to\infty}\{T^{(k)}\}$ exists since $\{T^{(k)}\}$ is monotone nondecreasing. Let this limit be T^*. Define a second subspace sequence $\{N^{(k)}\}$ by

$$\begin{aligned} N^{(k+1)} &= T^* + E(A^{-1}(N^{(k)} + \mathrm{Im} B) \cap \ker C), \\ N^{(0)} &= T^* + \mathrm{Im} E. \end{aligned}$$

$\{N^{(k)}\}$ is monotone nonircreasing and its limit is denoted by $\mathbf{N}^*$. Finally, define two important sequences $\{P^{(k)}\}$ and $\{S^{(k)}\}$ by

$$\begin{aligned} P^{(k+1)} &= \mathrm{Ker}\, C \cap A^{-1}(EP^{(k)} + \mathrm{Im} B), \; P^{(0)} = R^n, \\ S^{(k+1)} &= \mathrm{Ker}\, C \cap E^{-1}(AS^{(k)} + \mathrm{Im} B), \; S^{(0)} = R^n. \end{aligned}$$

Within at most n steps, recursion $\{P^{(k)}\}$ converge to $\mathbf{T}_1^*$, recursion $\{S^{(k)}\}$ converge to $\mathbf{T}_2^*$, exactly. We have that

Proposition 2.1 $\mathbf{N}^* = E\mathbf{T}_1^* + A\mathbf{T}_2^*$.

Proof It follows immediately from the definitions that $T^{(k)} = AS^{(k)}$ for all $k \ge 0$, Thus, we have $\mathbf{T}^* = A\mathbf{T}_2^*$, and therefore $N^{(0)} = A\mathbf{T}_2^* + EP^{(0)}$. Now assume that $N^{(k)} = A\mathbf{T}_2^* + EP^{(k)}$. Then

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$$
\begin{aligned}
N^{(k+1)} &= A\mathbf{T}_2^* + E(A^{-1}(N^{(k)} + \mathrm{Im}B) \cap \mathrm{Ker}C) \\
&= A\mathbf{T}_2^* + E\{A^{-1}(A\mathbf{T}_2^* + EP^{(k)} + \mathrm{Im}B) \cap \mathrm{Ker}C\} \\
&= A\mathbf{T}_2^* + E\{(\mathbf{T}_2^* + A^{-1}(EP^{(k)} + \mathrm{Im}B)) \cap \mathrm{Ker}C\} \\
&= A\mathbf{T}_2^* + E\{\mathbf{T}_2^* \cap \mathrm{Ker}C + A^{-1}(EP^{(k)} + \mathrm{Im}B) \cap \mathrm{Ker}C\} \quad (\mathbf{T}_2^* \subseteq \mathrm{Ker}C) \\
&= A\mathbf{T}_2^* + E\{\mathbf{T}_2^* + P^{(k+1)} = A\mathbf{T}_2^* + E\mathbf{T}_2^* + EP^{(k+1)} \\
&= A\mathbf{T}_2^* + EP^{(k+1)} \quad (E\mathbf{T}_2^* \subseteq A\mathbf{T}_2^*).
\end{aligned}
$$

This proves that $N^{(k+1)} = AT_2^* + EP^{(k+1)}$ for all $k \geq 0$. Consequently, we conclude that

$$
\mathbf{N}^* = A\mathbf{T}_2^* + E\mathbf{T}_1^*. \qquad \square
$$

We can now present our result as follows

Theorem 2.1 *If*

(a) $\mathbf{T}_2^* \cap \mathrm{Ker}E = \{0\}$,

(b) $\dim(E\mathbf{T}_2^* \cap \mathrm{Im}B) \leq \dim\{u \in \mathbf{T}_2^* : Au \in \mathrm{Im}B\}$,

(c) $Ex(0-) \in \mathbf{N}^*$,

then the disturbance rejection problem for singular systems is solvable via state feedback if and only if $\mathrm{Im}S \subseteq \mathbf{N}^* + \mathrm{Im}B$.

Proof Necessity. Taking the Laplacian transform of (1.3a) (1.3b) with initial coondition $Ex(0-)$, we get

$$
[sE - (A - BF)]x(s) = Ex(0-) + Sd(s), \tag{2.1a}
$$

$$
y(s) = Cx(s), \tag{2.1b}
$$

$x(s), y(s), d(s)$ are identified by the infinite sequences

$$
\begin{aligned}
&\{x_{-\mu}, x_{-\mu+1}, \ldots, x_0, x_1, \ldots\}, \\
&\{y_{-\mu}, y_{-\mu+1}, \ldots, y_0, y_1, \ldots\}, \\
&\{d_{-\mu-1}, d_{-\mu}, d_{-u-1}, \ldots, d_0, d_1, \ldots\}
\end{aligned}
$$

separately, defined by [9].

$$
x(s) = x_{-\mu}s^{\mu} + x_{-\mu+1}s^{\mu-1} + \ldots + x_{-1}s + x_0 + x_1s^{-1} + x_2s^{-2} + \ldots, \tag{2.2}
$$

$$
y(s) = y_{-\mu}s^{\mu} + y_{-\mu+1}s^{\mu-1} + \ldots + y_{-1}s + y_0 + y_1s^{-1} + y_2s^{-2} + \ldots, \tag{2.3}
$$

$$
d(s) = d_{-\mu-1}s^{\mu+1} + d_{-\mu}s^{\mu} + d_{-\mu+1}s^{\mu-1} + \ldots d_{-1}s + d_0 + d_1s^{-1} + d_2s^{-2} + \ldots, \tag{2.4}
$$

when no confusion is possible, we abuse the terminology and refer to $x(s), y(s), d(s)$ or to their laurent expansions as the trajectory, the output and the disturbance generated by the initial condition $Ex(0-)$.

Now, disturbance rejection demands for some F, for any $Ex(0-) \in N^*$, the Laurent expansions of $x(s), y(s), d(s)$ which satisfy (2.1a) (2.1b) satisfy the following equations for some μ:

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$$
\begin{array}{ll}
Ex_{-\mu} = Sd_{-\mu-1}, & Cx_{-\mu} = 0, \\
Ex_{-\mu-1} = (A - BF)x_{-\mu} + Sd_{-\mu}, & Cx_{-\mu-1} = 0, \\
\dots & \dots \\
Ex_{-1} = (A - BF)x_{-2} + Sd_{-2}, & Cx_{-1} = 0, \\
Ex_0 = (A - BF)x_{-1} + Sd_{-1}, & Cx_0 = 0, \\
Ex_1 = (A - BF)x_0 + Ex(0-) + Sd_0, & Cx_1 = 0, \\
Ex_2 = (A - BF)x_1 + Sd_1, & Cx_2 = 0, \\
\dots & \dots
\end{array}
$$

Note that $P^{(0)} = R^n$, from the first equation we get that

$$Sd_{-\mu-1} = Ex_{-\mu} = A(A^{-1}Ex_{-\mu}) \in A\mathbf{T}_1^* \subseteq E\mathbf{T}_1^* + \text{Im}B.$$

Form the second equation, we get

$$
\begin{aligned}
Sd_{-\mu} &= Ex_{-\mu-1} - (A - BF)X_{-\mu} \\
&= A(A^{-1}EX_{-\mu+1}) - E[E^{-1}(AX_{-\mu} - BFx_\mu)] \\
&\in A\mathbf{T}_1^* + E\mathbf{T}_2^* \subseteq E\mathbf{T}_1^* + A\mathbf{T}_2^* + \text{Im}B.
\end{aligned}
$$

By the same way, we know $Sd_k \in E\mathbf{T}_1^* + A\mathbf{T}_2^* + \text{Im}B$ *for all* $k \geq -\mu - 1$, so

$$\text{Im}S \subseteq E\mathbf{T}_1^* + A\mathbf{T}_2^* + \text{Im}B.$$

Sufficiency. Suppose $\mathbf{T}_1^*$, $\mathbf{T}_2^*$ satisfy (a) (b), from Theorem 1.1, we can find $F \in \mathbf{T}(\mathbf{T}_1^*) \cap \mathbf{T}(\mathbf{T}_2^*)$ such that $(A - BF)\mathbf{T}_1^* \subseteq E\mathbf{T}_1^*$, $E\mathbf{T}_2^* \subseteq (A - BF)\mathbf{T}_2^*$. and $A - BF - \lambda E$ has linearly independent columns for some complex number λ. By theorem 1 of [3], from the matrix $A - BF - \lambda E$ having linearly independent columns for some complex number λ, we can reach the uniqueness of solution to the closed-loop system (1.3a). thus it is sufficient to consider existence of solutions since $\text{Im}S \subseteq E\mathbf{T}_1^* + A\mathbf{T}_2^* + \text{Im}B$, according to Theorem 3.2 of [1], we know (1.3a) (1.3b) has a solution $x(t)$, furthermore $x(t) \in \mathbf{T}_1^* + \mathbf{T}_2^* \subseteq \text{Ker}\, C$ which means $y(t) = Cx(t) = 0$. This completes the proof.

The proof of following lemma is obvious and is omitted.

Lemma *Given* S_1, B *and a subspace* $\mathcal{V} \subseteq X$, *there exists* $F_1: D \to X$ *such that*

$$\text{Im}(S_1 - BF_1) \subseteq \text{Im}B + \mathcal{V}$$

if and only if $\text{Im}S_1 \subseteq \text{Im}B + \mathcal{V}$.

According to the above lemma and Theorem 2.1, we can obtain our final result easily.

Theorem 2.2 *If*

(*a*) $\mathbf{T}_1^* \cap \text{Ker}E = \{0\}$,

(*b*) $\dim(E\mathbf{T}_2^* \cap \text{Im}B) \leq \dim\{u \in \mathbf{T}_2^* : Au \in \text{Im}B\}$,

(*c*) $Ex(0-) \in \mathbf{N}^*$,

then the disturbance rejection for singular systems via linear state feedback and linear feedfor-

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ward control of the disturbances is solvable if and only if $\mathrm{Im}\, S_1 \subseteq \mathbf{N}^* + \mathrm{Im} B$.

III Conclusion

We have provided necessary and sufficent condition for the solvability of disturbance rejection for singular systems. We have taken care to ensure that the resulting closed-loop system has smooth solutions for a wide class of disturbance functions and initial conditions and that. When a solution exists, it is unique.

Our mode of argument has been in the spirit of Wonham[2] and the structure of our proof clearly resembles that of the corresponding result in the state space case. We have found the usual presentation, in terms of the properties of uniquely edtermined maximal elements of certain collections of subspaces, by means of some subspace recursions. This means that our result is constructive and algorithmic.

Throughout this paper we have had in mind a linear system described by a mixture of algebraic and differential equations, the extension of these results to the discrete-time case should not present any major difficulty.

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奇异系统的干扰解耦

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摘　要

本文针对线性的不变奇异系统, 讨论了利用一般状态反馈及前馈控制的组合来达到其干扰解耦的条件. 文中构造性地给出了这一问题可解的充分必要条件.

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