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Inverses of Regular Strong Endomorphisms of Graphs*

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Abstract In this paper we explicitly describe all the commuting pseudo-inverses of a completely regular strong endomorphism of a graph from a viewpoint of combinatorics. The number of them is also given. In addition, a strong endomorphism of a graph, whose commuting pseudo-inverse set coincides with its pseudo-inverse set, is identified.

Keywords strong endomorphism, (commuting) pseudo-inverse, monoid, graph.

Classification AMS(1991) 05C25/CCL O157.5

This paper is a continuation of [1], in which pseudo-inverses of a strong endomorphism of a graph were investigated, in particular, the characterization and the enumeration of them were explicitly described. The main purpose of this paper is to characterize and to enumerate commuting pseudo-inverses of a completely regular strong endomorphism of a graph. It turns out that pseudo-inverses and commuting pseudo-inverses of a strong graph endomorphism indeed possess distinctly comparable features from the viewpoint of combinatorics. In addition, a strong endomorphism of a graph, whose commuting pseudo-inverse set coincides with its pseudo-inverse set is identified. These results reveal the interconnection between the combinatorial structure of a graph and the algebraic structure of its corresponding monoid. Readers are referred to [2] and [3] for more background information in this line.

Let a be an element of a semigroup S. If there exists $x \in S$ such that $axa = a$, then a is said to be regular and x is called a pseudo-inverse of a; moreover, if there exists $x \in S$ such that $axa = a$ and $ax = xa$, then a is said to be completely regular and x is called a commuting pseudo-inverse of a (cf. [4] or [5]). A semigroup is said to be regular (or completely regular) if all its elements are regular (or completely regular). In [2] and [6], it is proved that the monoid of strong endomorphisms of a graph is always regular. In [7], completely regular endomorphisms of a graph are characterized.

Only finite undirected graphs without loops and multiple edges are considered in this paper. We denote by $V(G)$ (or just G) and $E(G)$ the vertex set and edge set of a graph G respectively. For definitions of endomorphism, strong endomorphism and automorphism of a graph, readers are referred to [1] or [2]. By $\mathrm{End}(G)$, $s\mathrm{End}(G)$ and $\mathrm{Aut}(G)$ we denote the set of endomorphisms, strong endomorphisms and automorphisms of graph G respectively. Apparently, $\mathrm{Aut}(G) \subseteq s\mathrm{End}(G) \subseteq \mathrm{End}(G)$. It is well-known that $\mathrm{End}(G)$ and $s\mathrm{End}(G)$ are monoids and that $\mathrm{Aut}(G)$ is a group with respect to the composition of mappings.

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Let $f \in s\mathrm{End}(G)$. We denote by S_f the induced subgraph of G with $V(S_f) = f(V(G))$, by ρ_f the equuivalence relation on $V(G)$ induced by f, i.e., for any $a, b \in V(G)$, $(a, b) \in \rho_f$ if and only if $f(a) = f(b)$. For $x \in G$ and $A \subseteq V(G)$, $f^{-1}(x) := \{a \in G \mid f(a) = x\}$ and $f^{-1}(A) := \cup_{x \in A} f^{-1}(x)$. For a vertex $a \in G$, we put $N(a) = \{b \in G \mid \{a, b\} \in E(G)\}$ (the neighbourhood of a in G). The relation τ on $V(G)$ is defined by the rule that $(a, b) \in \tau$ if and only if $N(a) = N(b)$. Clearly, the relation τ is an equivalence relation on $V(G)$. By $[a]_\tau$ we denote the equivalence class of the vertex a of graph G with respect to the relation τ. Fundamental definitions and notations not defined in this paper should be referred to [8] and [9]. $\mathrm{CPI}(f)$ and $\mathrm{PI}(f)$ denote the sets of commuting pseudo-inverses and pseudo-inverses of f respectively, i.e., $\mathrm{PI}(f) = \{g \in s\mathrm{End}(G) \mid fgf = f\}$; $\mathrm{CPI}(f) = \{g \in s\mathrm{End}(G) \mid fgf = f; gf = fg\}$. Evidently, $\mathrm{CPI(f)} \subseteq \mathrm{PI(f)}$.

Theorem 1 [7, Theorem 2.7] *Let G be a graph and let $f \in s\mathrm{End}(G)$. Then f is completely regular if and only if for any $x \in G$, $|[x]_{\rho_f} \cap S_f| = 1$.*

Lemma 2 *Let G be a graph and let $f \in s\mathrm{End}(G)$. If f is completely regular, then for any $x \in S_f$, $|f^{-1}(x) \cap S_f| = 1$.*

Proof Since $x \in S_f$, there exists $x' \in G$ such that $f(x') = x$. As f is completely regular, using Theorem 1 we have $|[x']_{\rho_f} \cap S_f| = 1$. It is routine to check that $f^{-1}(f(x')) = [x']_{\rho_f}$ and so $f^{-1}(x) = [x']_{\rho_f}$. Hence, $|f^{-1}(x) \cap S_f| = 1$. □

Definition 3 *Let G be a graph and let $f \in s\mathrm{End}(G)$ such that f is completely regular. Define a mapping $g: V(G) \rightarrow V(G)$ by the following rule:*

$$(3.1) \qquad g(x) \in \begin{cases} f^{-1}(x) \cap S_f & \text{if } x \in S_f; \\ f^{-1}([x]_{\rho_f} \cap S_f) & \text{if } x \in V(G) \setminus S_f. \end{cases}$$

(Just as in [1], by, e.g. $g(x) \in f^{-1}(x) \cap S_f$, we mean that "select a vertex $y \in f^{-1}(x) \cap S_f$ and set $g(x) = y$.")

Remark 4 *Under the hypothesis of Definition 3,*

(1) *the mapping g defined by (3.1) is well defined;*

(2) *for any $x \in G$, $[x]_{\rho_f} \cap S_f = \{gf(x)\}$.*

Proof (1) By Lemma 2, for any $x \in S_f$, $f^{-1}(x) \cap S_f \neq \emptyset$ and so $g(x)$ is correctly defined. Since f is completely regular, so by Theorem 1 for any $x \in G$, $[x]_{\rho_f} \cap S_f \neq \emptyset$. Evidently, $[x]_{\rho_f} \cap S_f \subseteq V(S_f)$, so $f^{-1}([x]_{\rho_f} \cap S_f) \neq \emptyset$. Hence, for $x \in V(G) \setminus S_f$, $g(x)$ is also correctly defined. This justifies the conclusion.

(2) Let $x \in G$. Since $f(x) \in S_f$, by (3.1) $gf(x) = g(f(x)) \in f^{-1}(f(x)) \cap S_f$. Since $f^{-1}(f(x)) = [x]_{\rho_f}$, $gf(x) \in [x]_{\rho_f} \cap S_f$. As f is completely regular, by Theorem 1 $|[x]_{\rho_f} \cap S_f| = 1$ and so $[x]_{\rho_f} \cap S_f = \{gf(x)\}$. □

Lemma 5 [3, Lemma 6] *Let G be a graph and let $f \in s\mathrm{End}(G)$. Then, for any $x, y \in G$, $N(f(x)) = N(f(y))$ if and only if $N(x) = N(y)$.*

Lemma 6 *Let G be a graph and let $f \in s\mathrm{End}(G)$. Then, for any $x \in G$, $[x]_{\rho_f} \subseteq [x]_\tau$*

Proof Let $a \in [x]_{\rho_f}$. Then $f(a) = f(x)$ and so $N(f(a)) = N(f(x))$. Thus, by Lemma 5, $N(a) = N(x)$, which means that $a \in [x]_\tau$. □

Lemma 7 [1, Theorem 2.8] *Let G be a graph and let $f \in sEnd(G)$. Let g be a mapping from $V(G)$ to itself. Then $g \in PI(f)$ if and only if for any $x \in S_f$, $g(x) \in f^{-1}(x)$ and for any $x \in V(G) \setminus S_f$, $g(x) \in f^{-1}([x]_\tau \cap S_f)$.*

Lemma 8 *Under the hypothesis of Definition 3, $g \in PI(f)$.*

Proof By Remark 4, g is well defined. For any $x \in S_f$, by (3.1), $g(x) \in f^{-1}(x) \cap S_f \subseteq f^{-1}(x)$. Using Lemma 6, we have $[x]_{\rho_f} \cap S_f \subseteq [x]_\tau \cap S_f$ for any $x \in G$. Thus, $f^{-1}([x]_{\rho_f} \cap S_f) \subseteq f^{-1}([x]_\tau \cap S_f)$. Hence, for any $x \in V(G) \setminus S_f$, $g(x) \in f^{-1}([x]_\tau \cap S_f)$. So, from Lemma 7 it follows that $g \in PI(f)$. □

Proposition 9 *Under the hypothesis of Definition 3, $g \in CPI(f)$.*

Proof By Lemma 8, it remains to prove that $fg = gf$. Let $x \in S_f$. Since f is completely regular, by Lemma 2 $f^{-1}(x) \cap S_f = \{y\}$ for some $y \in G$. So by (3.1), we have $g(x) \in \{y\}$, i.e. $g(x) = y$. Since $y \in f^{-1}(x)$, $f(y) = x$. Hence, $fg(x) = f(g(x)) = f(y) = x$. On the other hand, since $x \in S_f$, $x \in [x]_{\rho_f} \cap S_f$. Then, using Remark 4(2) we have $x = gf(x)$. This verifies that $fg(x) = gf(x)$ for any $x \in S_f$.

Now, let $x \in V(G) \setminus S_f$. Then according to (3.1), $g(x) \in f^{-1}([x]_{\rho_f} \cap S_f)$ and so $fg(x) \in [x]_{\rho_f} \cap S_f$. Hence, by Remark 4(2) we also have $fg(x) = gf(x)$ for any $x \in V(G) \setminus S_f$. Consequently, $fg = gf$. This completes the proof. □

Proposition 10 *Let G be a graph and let $f \in sEnd(G)$ such that f is completely regular. If $g \in CPI(f)$, then for any $x \in S_f$, $g(x) \in f^{-1}(x) \cap S_f$; for any $x \in V(G) \setminus S_f$, $g(x) \in f^{-1}([x]_{\rho_f} \cap S_f)$.*

Proof Let $x \in S_f$. Since $g \in CPI(f)$, $g \in PI(f)$. Then by Lemma 7, $g(x) \in f^{-1}(x)$. Since f is completely regular, by Lemma 2 $f^{-1}(x) \cap S_f = \{y\}$ for some $y \in G$. Thus $f(y) = x$ and so $gf(y) = g(x)$. Since $g \in CPI(f)$, $fg = gf$. So, it follows that $fg(y) = g(x)$. Evidently, $fg(y) \in S_f$ and so $g(x) \in S_f$. Hence, we have that $g(x) \in f^{-1}(x) \cap S_f$ for any $x \in S_f$.

Now, let $x \in V(G) \setminus S_f$. Since $g \in CPI(f)$, we have $fgf(x) = f(x)$ and $fg(x) = gf(x)$. So, $ffg(x) = f(x)$, which means that $fg(x) \in f^{-1}(f(x))$. Notice that $f^{-1}(f(x)) = [x]_{\rho_f}$ and that $fg(x) \in S_f$. Hence, $fg(x) \in [x]_{\rho_f} \cap S_f$ so that $g(x) \in f^{-1}([x]_{\rho_f} \cap S_f)$. This completes the proof. □

Now, combining Propositions 9 and 10, we obtain the characterization of the commuting pseudo-inverses of a completely regular strong endomorphism f of a graph G as follows:

Theorem 11 *Let G be a graph and let $f \in sEnd(G)$ such that f is completely regular. Then the following two statements are equivalent:*

(1) $g \in CPI(f)$;

(2) *g is a mapping from $V(G)$ to itself such that (3.1) is satisfied.*

Recalling the characterization of pseudo-inverses of a regular strong endomorphism of a graph (cf. [1, Theorem 2.8]), we have

$$g \in \mathrm{PI}(f) \Leftrightarrow g(x) \in \begin{cases} f^{-1}(x) & \text{if } x \in S_f; \\ f^{-1}([x]_\tau \cap S_f) & \text{if } x \in V(G)\setminus S_f. \end{cases}$$

$$g \in \mathrm{CPI}(f) \Leftrightarrow g(x) \in \begin{cases} f^{-1}(x) \cap S_f & \text{if } x \in S_f; \\ f^{-1}([x]_{\rho_f} \cap S_f) & \text{if } x \in V(G)\setminus S_f. \end{cases}$$

From the above formulations we can see that the two purely algebraic concepts, namely, pseudo-inverses and commuting pseudo-inverses of a strong endomorphism, indeed possess distinctly comparable combinatorial features

Corollary 12 *Let G be a graph and let $f \in s\mathrm{End}(G)$ such that f is completely regular. Then*

$$|\mathrm{CPI}(f)| = \begin{cases} 1 & \text{if } f \in \mathrm{Aut}(G); \\ \prod_{x \in V(G)\setminus S_f} |f^{-1}([x]_{\rho_f} \cap S_f)| & \text{otherwise.} \end{cases}$$

Proof By Lemma 2, $|f^{-1}(x) \cap S_f| = 1$ for any $x \in S_f$. It is well known that $\mathrm{Aut}(G)$ is a group and so each $f \in \mathrm{Aut}(G)$ is completely regular. Also notice that $f \in \mathrm{Aut}(G)$ if and only if $V(G)\setminus S_f = \varnothing$. Hence, the conclusion follows directly from Theorem 11. □

Remark 13 *In general, if $f \in s\mathrm{End}(G)$ such that $|\mathrm{CPI}(f)| = 1$, it is not necessiarily that $f \in \mathrm{Aut}(G)$. For example, observe the graph G in Fig. 1 and $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 2 & 2 \end{pmatrix} \in s\mathrm{End}(G)$ (cf. [1, Example 4. 4]).*

Notice that $S_f = \{1234\}$ and $V(G)\setminus S_f = \{5\}$. By Theorem 11, it is routine to check that $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 2 & 1 & 1 \end{pmatrix}$ is the unique commuting pseudo-inverse of f. However, it is clear that $f \notin \mathrm{Aut}(G)$.

Example 14 Here we still take the graph and the strong endomorphism f in [1, Example 2. 10] as an example.

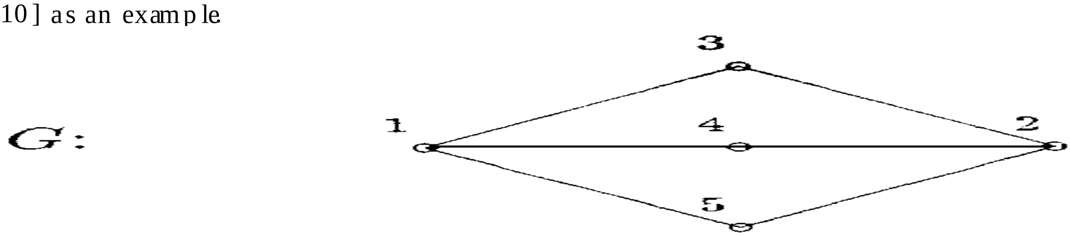

Fig. 1

It is known that $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 5 & 3 & 3 \end{pmatrix} \in s\,\mathrm{End}(G)$. Using Theorem 1, we can easily check that f is completely regular.

$S_f = \{235\}$, $f^{-1}(2) \cap S_f = \{12\} \cap \{235\} = \{2\}$, $f^{-1}(3) \cap S_f = \{45\} \cap \{235\} = \{5\}$, $f^{-1}(5) \cap S_f = \{3\} \cap \{235\} = \{3\}$;

$V(G)\setminus S_f = \{14\}$, $f^{-1}([1]_{\rho_f} \cap S_f) = f^{-1}(\{12\} \cap \{235\}) = f^{-1}(\{2\}) = \{12\}$, $f^{-1}([4]_{\rho_f} \cap S_f) = f^{-1}(\{45\} \cap \{235\}) = f^{-1}(\{5\}) = \{3\}$.

So, by Theorem 11 and Corollary 12 we see that $CPI(f)$ exactly contains the following two elements, i.e.

$$CPI(f) = \{f, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 5 & 3 & 3 \end{pmatrix}\}.$$

In [1, Example 2.10], we deduced $|PI(f)| = 24$, and now $|CPI(f)| = 2$.

Now, at the end of this paper, we turn to a natural question: for which $f \in sEnd(G)$, $PI(f) = CPI(f)$. We will see that such strong endomorphisms are none other than automorphisms. We first list known statements as lemmas:

Lemma 15 [1. Remark 2.11] *Let G be a graph and let $f \in sEnd(G)$. Then $f \in Aut(G)$ if and only if $|PI(f)| = 1$.*

Lemma 16 [6. Lemma 2.2] *Let G be a graph and let $f \in sEnd(G)$. Then for any $x \in G$, $[x]_\tau \cap S_f \neq \emptyset$.*

Theorem 17 *Let G be a graph and let $f \in sEnd(G)$. Then $PI(f) = CPI(f)$ if and only if $f \in Aut(G)$.*

Proof Sufficiency. Let $f \in Aut(G)$. Notice that $CPI(f) \subseteq PI(f)$. Then from Corollary 12 and Lemma 15, it follows directly that $PI(f) = CPI(f)$.

Necessity. Now, we assume that $f \notin Aut(G)$ and we want to prove that $PI(f) \neq CPI(f)$. If f is not completely regular, then $CPI(f) = \emptyset$ and so $PI(f) \neq CPI(f)$ because $PI(f) \neq \emptyset$. So, we suppose f is completely regular.

Since $f \notin Aut(G)$, there exists a vertex $a \in G$ with $a \notin S_f$. Define a mapping $g: V(G) \to V(G)$ by the following rule:

$$g(f(a)) = a;$$
$$g(x) \in f^{-1}(x) \text{ if } x \in S_f \setminus \{f(a)\};$$
$$g(x) \in f^{-1}([x]_\tau \cap S_f) \text{ if } x \in V(G) \setminus S_f.$$

Considering that for any $x \in G$, $[x]_\tau \cap S_f \neq \emptyset$ by Lemma 16 and that $[x]_\tau \cap S_f \subseteq S_f$, we have $f^{-1}([x]_\tau \cap S_f) \neq \emptyset$. For $x \in S_f \setminus \{f(a)\}$, obviously $f^{-1}(x) \neq \emptyset$. Hence, the mapping g is well defined. We further prove that $g \in PI(f)$ but $g \notin CPI(f)$.

Notice that $f(a) \in S_f$ and $g(f(a)) = a \in f^{-1}(f(a))$. So, for any $x \in S_f$, $g(x) \in f^{-1}(x)$. Thus, by Lemma 7, $g \in PI(f)$. However, since $a \notin S_f$, $a \notin f^{-1}(f(a)) \cap S_f$, i.e. $g(f(a)) \notin f^{-1}(f(a)) \cap S_f$. So, put $y = f(a)$ and we see that $y \in S_f$ but $g(y) \notin f^{-1}(y) \cap S_f$. Recall that we have supposed that f is completely regular. Thus, from Theorem 11 it follows immediately that $g \notin CPI(f)$. Consequently, $PI(f) \neq CPI(f)$. This completes the proof. □

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图的正则强自态的逆

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摘　　要

本文从组合角度明确刻画了图的完全正则强自同态的交换伪逆, 同时给出它们的计数公式