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A Generalization of KKM Theorem *

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Abstract In this paper we introduce the concepts of Z-space and Z-KKM mapping. generalize famous F-KKM theorem and other versions of KKM theorem.

Keywords KKM-theorem. Z-space. Z-KKM mapping.

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1. Introduction and Preliminaries

In 1961, Fan K.[11] generalized the KKM-theorem[10] to the infinite dimensional case. Since then many versions of F-KKM theorem were obtained [Refs. 2,3,4,6,9].

In section 1 we introduce the Z-space concept and, in section 2 we establish a general version of KKM-theorem, which is the unification and generalization of many versions of KKM-theorem, and obtain a new version of KKM-theorem: Theorem 2.5. In section 3 we discuss some spacial cases.

Throughout this paper **N** denotes the set of natural number. For each $n \in \mathbf{N}$, set $[n] = \{1, \dots, n\}$. By $\mathrm{F}([n])$ we denote the family of nonempty subsets of $[n]$.

Let X be a nonempty set, for each $n \in \mathbf{N}$, set

$$\mathrm{F}_n(X) = \{A \subset X : A \text{ is a subset of } X \text{ with } n \text{ elements}\},$$

$$\mathrm{F}(X) = \bigcup_1^{\infty} \mathrm{F}_n(X).$$

Let $A = \{x_1, \dots, x_n\} \in \mathrm{F}_n(X)$, for $I \in \mathrm{F}([n])$, set $A_I = \{x_i : i \in I\}$.

For $n \in \mathbf{N}$, we denote, by $\Delta^{(n-1)} = e_1 \cdots e_n$, the standard $(n-1)$-simplex. For $I \in \mathrm{F}([n])$, $\Delta_I^{(n-1)} = \mathrm{co}\{e_i : i \in I\}$ is a sub-simplex of $\Delta^{(n-1)}$.

Let X, Y be two topological spaces by $C(X, Y)$ we denote the family of continuous mappings from X to Y.

Definition 1.1 *Let X be a topological space and $\{F_{A.B}\}$ be the family of non-empty subsets of X, labeled by all $A \in \mathrm{F}(X)$ and $B \in \mathrm{F}(A)$. $(X, \{F_{A.B}\})$ is said a Z-space,*

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if and only if for each $n \in \mathbf{N}$ *and each* $A = \{x_1, \ldots, x_n\} \in \mathrm{F}_n(X)$, *there exists a* $f_A \in \mathrm{C}(\Delta^{(n-1)}, X)$ *such that*

$$f_A(\Delta_I^{(n-1)}) \subset F_{A.A_I}, \quad \text{for all } I \in \mathrm{F}([n]) \ .$$

Remark 1.1 H-space $(X, \{\Gamma_A\})$ [see 1] is a special example of Z-space. In fact for each $A \in \mathrm{F}(X)$, and each $B \in \mathrm{F}(A)$, let $F_{A.B} = \Gamma_B$. Then for each $n \in \mathbf{N}$, and each $A = \{x_1, \ldots, x_n\} \in \mathrm{F}_n(X)$, by [1], there exists a continuous mapping $f_A : \Delta^{(n-1)} \to X$, such that $f_A(\Delta_I^{(n-1)}) \subset \Gamma_{A_I}$, for all $I \in \mathrm{F}([n])$. Therefore

$$f_A(\Delta_I^{(n-1)}) \subset \Gamma_{A_I} = F_{A.A_I} \quad \text{for all } I \in \mathrm{F}([n]).$$

Thus by [2] Hausdorff topological vector space, convex space [8], contractible space, and pseudo-convex space [2] are all the special cases of Z-space.

Let $(X, \{F_{A.B}\})$ be a Z-space. A subset $D \subset X$ is called Z-convex relative to subset $C \subset X$, If for each $B \in \mathrm{F}(C)$ and each $A \in \mathrm{F}(X)$ with $B \subset A$, it follows $F_{A.B} \subset D$. When $C = D$, then D is called Z-convex.

Let $(X, \{F_{A.B}\})$ be a Z space and D is a nonempty subset of X. Set

$$\mathrm{co}(D) = \cap\{C : \ C \text{ is a } Z\text{-convex subset of } X \text{ and } D \subset C\},$$

$\mathrm{co}(D)$ is called Z-convex hull of D. It is clear that $\mathrm{co}(D)$ is a Z-convex subset of X and $D \subset \mathrm{co}(D)$.

Definition 1.2 *Let* W *be a nonempty set,* Y *a topological space and* $G : W \to 2^Y$ *a set-valued mapping. If for any* $n \in \mathbf{N}$ *and any* $E = \{w_1, \ldots, w_n\} \in \mathrm{F}_n(W)$, *there exist a* Z*-space* $(X, \{F_{A.B}\})$, *a set* $A = \{x_1, \ldots, x_n\} \in \mathrm{F}_n(X)$, *and a* $s \in \mathrm{C}(X, Y)$ *such that*

$$s(F_{A.A_I}) \subset G(E_I) \quad \text{for all } I \in \mathrm{F}([n]), \tag{1}$$

then G *is called a Z-KKM mapping.*

Remark 1.2 Definition 1.2 is the most general concept and it contains KKM mapping, H-KKM mapping [2], generalized KKM mapping [3, 4], and GH-KKM mapping [5] as its special cases.

Definition 1.3 *Let* X *be a topological space,* D *a subset of* X. D *is said a compactly open (respectively compactly closed) subset of* X, *if, for any compact subset* C *of* X, $D \cap C$ *is a relatively open (respectively relatively closed) subset of* C.

Obviously if D is a compactly open (compactly closed) subset of X, then $X \backslash D$ is a compactly closed (compactly open) subset of X. Also if f is a continuous mapping from topological space X to topological space Y and D is a compactly open (compactly closed) subset of Y, then $f^{-1}(D)$ is a compactly open (compactly closed) subset of X.

Lemma *Let* $(X, \{F_{A.B}\})$ *be a* Z*-space, and* $M_1, \ldots, M_n$ *be* n *compactly closed subsets of* X *such that* $\cup_{i=1}^n M_i = X$. *Then for any* $A = \{x_1, \ldots, x_n\} \in \mathrm{F}_n(X)$, *there exists* $I \in \mathrm{F}([n])$ *such that* $F_{A.A_I} \cap (\cap_{i \in I} M_i) \neq \emptyset$.

Proof For $A = \{x_1, \ldots, x_n\} \in \mathrm{F}_n(X)$, by definition of Z-space, take a mapping $f_A \in \mathrm{C}(\Delta^{(n-1)}, X)$ such that $f_A(\Delta_I^{(n-1)}) \subset F_{A.A_I}$ for all $I \in \mathrm{F}([n])$. For $u \in \Delta^{(n-1)}$, set

$$I(u) = \{i \in [n] : f_A(u) \in M_i\} \in \mathrm{F}([n]),$$
$$S(u) = \Delta_{I(u)}^{(n-1)}.$$

We have $I(u) \neq \emptyset$, and then $S(u) \neq \emptyset$. By compactly closedness of M_i we can prove that $U = \Delta^{(n-1)} \backslash f_A^{-1}(\cup_{i \notin I(u)} M_i)$ is a open neighborhood of u in $\Delta^{(n-1)}$. If $u' \in U$, then $I(u') \subset I(u)$. So $S(u') \subset S(u)$. Therefore mapping $S : \Delta^{(n-1)} \to 2^{\Delta^{(n-1)}}$ is upper semi-continuous with nonempty compact convex value. From Kakutani's fixed point theorem there exists a $u_0 \in \Delta^{(n-1)}$ such that $u_0 \in S(u_0) = \Delta_{I(u_0)}^{(n-1)}$. From the definition of $I(u_0)$, $f_A(u_0) \in \cap_{i \in I(u_0)} M_i$. On the other hand, set $I = I(u_0)$, $f_A(u_0) \in f_A(\Delta_I^{(n-1)}) \subset F_{A.A_I}$. So

$$F_{A.A_I} \cap (\cap_{i \in I} M_i) \neq \emptyset.$$

2. The main results

Theorem 2.1 *Let X be a nonempty set, Y a topological space. Suppose that $G : W \to 2^Y$ is a Z-KKM mapping, and one of the following conditions is hold:*

(i) for each $w \in W$, $G(w)$ is compactly open in Y.

(ii) for each $w \in W$, $G(w)$ is compactly closed in Y.

Then the family of $\{G(w) : w \in W\}$ of sets has the finite intersection property. Moreover, if we add the following condition to condition (ii),

(*) there exists a $w_0 \in W$ such that $G(w_0)$ is a compact set then $\cap_{w \in W} G(w) \neq \emptyset$.

Proof (i) Suppose that for each $w \in W$, $G(w)$ is compactly open.

By contradiction suppose that $\{G(w) : w \in W\}$ has not the finite intersection property. Then there exist $n \in \mathbf{N}$ and $E = \{w_1, \ldots, w_n\} \in \mathrm{F}_n(W)$, such that

$$\bigcap_{i=1}^{n} G(w_i) = \emptyset. \tag{2}$$

Because G is a Z-KKM mapping, there exist Z-space $(X, \{F_{A.B}\})$, $A = \{x_1, \ldots, x_n\} \in \mathrm{F}_n(X)$, and $s \in \mathrm{C}(X, Y)$, such that

$$s(F_{A.A_I}) \subset G(E_I) \quad \text{for all } I \in \mathrm{F}([n]).$$

From (2) we have $\cap_{i=1}^n s^{-1}(G(w_i)) = \emptyset$. Let $H_i = X \backslash s^{-1}(G(w_i))$, $i = 1, \ldots, n$. Then H_i are compactly closed subset of X and $\cup_{i=1}^n H_i = X$. By Lemma there exists an $I' \in \mathrm{F}([n])$, such that

$$F_{A.A_{I'}} \bigcap (\bigcap_{i \in I'} H_i) \neq \emptyset. \tag{3}$$

On the other hand since $s(F_{A.A_{I'}}) \subset G(E_{I'})$, $F_{A.A_{I'}} \cap (X \backslash s^{-1}(G(E_{I'}))) = \emptyset$, i.e.,

$$F_{A.A_{I'}} \bigcap (\bigcap_{i \in I'} H_i) = \emptyset. \tag{4}$$

This contradicts (3). Hence $\{G(w) : w \in W\}$ has the finite intersection property.

(ii) Suppose that for each $w \in W$, $G(w)$ is compactly closed.

For any $n \in \mathbf{N}$ and any $E = \{w_1, \dots, w_n\} \in \mathrm{F}_n(W)$, Because G is a Z-KKM mapping, there exist Z-space $(X, \{F_{A.B}\})$, $A = \{x_1, \dots, x_n\} \in \mathrm{F}_n(X)$, and $s \in \mathrm{C}(X, Y)$, such that $s(F_{A.A_I}) \subset G(E_I)$ for all $I \in \mathrm{F}([n])$. From the definition of Z-space there exists a $f_A \in \mathrm{C}(\Delta^{(n-1)}, X)$ such that

$$f_A(\Delta_I^{(n-1)}) \subset F_{A.A_I} \quad \text{for all } I \in \mathrm{F}([n]),$$

so $s(f_A(\Delta_I^{(n-1)})) \subset G(E_I)$ for all $I \in \mathrm{F}([n])$, and

$$\Delta_I^{(n-1)} \subset f_A^{-1}(s^{-1}(G(E_I))) = \bigcup_{i \in I} f_A^{-1}(s^{-1}(G(w_i))) \quad \text{for all } I \in \mathrm{F}([n]).$$

Since $G(w_i)$ is compactly closed $f_A^{-1}(s^{-1}(G(w_i)))$ is closed in $\Delta^{(n-1)}$. By well-known KKM-theorem there exists a $\bar{u} \in \Delta^{(n-1)}$, such that $\bar{u} \in \cap_{i=1}^n f_A^{-1}(s^{-1}(G(w_i)))$, that is $s(f_A(\bar{u})) \in \cap_{i=1}^n G(w_i)$. Therefore $\{G(w) : w \in W\}$ has the finite intersection property. Moreover when $G(w_0)$ is compact, for each $w \in W$, $G(w) \cap G(w_0)$ is nonempty compact. Hence

$$\bigcap_{w \in W} G(w) = \bigcap_{w \in W} (G(w) \bigcap G(w_0)) \neq \emptyset.$$

Remark 2.1 Theorem 2.1 unifies and generalizes results those in [2–6, 8, 9].

Theorem 2.2 *Let W be a nonempty set, Y a topological space. Suppose that $G : W \to 2^Y$ is a Z-KKM mapping. If*

(i) for each $w \in W$, $G(w)$ is compactly closed in Y,

(ii) there exists $W_0 \subset W$, such that $\cap_{w \in W_0} G(w)$ is compact,

(iii) for any $E_0 \in \mathrm{F}(W)$, there exists a nonempty compact subset Y_0 of Y such that for each $E = \{w_1, \dots, w_n\} \in \mathrm{F}(W_0 \cup E_0)$,

$$s(F_{A.A_I}) \subset Y_0 \quad \textit{for all } I \in \mathrm{F}([n]),$$

where $(X, \{F_{A.B}\}), A = \{x_1, \dots, x_n\}$ and $s : X \to Y$ as in Definition 1.2.

Then $\bigcap_{w \in W} G(w) \neq \emptyset$.

Proof Given $E_0 \in \mathrm{F}(W)$, from (iii) take nonempty compact $Y_0 \subset Y$. Define $\tilde{G} : W_0 \cup E_0 \to 2^Y$ by $\tilde{G}(w) = G(w) \cap Y_0$ for all $w \in W_0 \cup E_0$. For each $E = \{w_1, \dots, w_n\} \in \mathrm{F}_n(W_0 \cup E_0) \subset \mathrm{F}_n(W)$, by assumption that G is a Z-KKM mapping and (iii) there exist a Z-space $(X, \{F_{A.B}\})$, $A = \{x_1, \dots, x_n\} \in \mathrm{F}_n(X)$ and $s \in \mathrm{C}(X, Y)$ such that

$$s(F_{A.A_I}) \subset Y_0 \bigcap G(E_I) = \tilde{G}(E_I) \quad \text{for all } I \in \mathrm{F}([n]).$$

This shows that $\tilde{G}$ is a Z-KKM mapping. From (i) and Y_0 is compact we have $\tilde{G}(w)$ is compact. By Theorem 2.1

$$\bigcap_{w \in W_0 \cup E_0} \tilde{G}(w) \neq \emptyset. \tag{5}$$

Let $K = \cap_{w\in W_0} G(w)$, by (ii) K is compact. And, from (5),

$$\bigcap_{w\in E_0}(K\cap G(w)) \supset \bigcap_{w\in W_0\cup E_0} G(w) \supset \bigcap_{w\in W_0\cup E_0} \tilde{G}(w) \neq \emptyset.$$

Thus we have proved that family $\{K\cap G(w) : w\in W\}$ has the finite intersection property. By (i) and (ii) $K\cap G(w)$ is compact. Therefore

$$\bigcap_{w\in W} G(w) = \bigcap_{w\in W}(K\cap G(w)) \neq \emptyset.$$

Theorem 2.3 *Let W be a nonempty set, Y a topological space. Suppose that $H : W \to Y$ satisfies*

(i) for each $w\in W$, $H(w)$ is compactly open,

(ii) there exists a $w_0\in W$, $Y\backslash H(w_0)$ is compact,

(iii) $H(W) = Y$.

Then there exist $n\in \mathbf{N}$ and $E = \{w_1,\ldots,w_n\}\in \mathrm{F}_n(W)$ such that for any Z-space $(X,\{F_{A.B}\})$, any $A = \{x_1,\ldots,x_n\}\in \mathrm{F}_n(X)$ and any $s\in \mathrm{C}(X,Y)$, there exist $I\in \mathrm{F}([n])$ and $x_0\in F_{A.A_I}$ with $s(x_0)\in \cap_{w\in E_I} H(w)$.

Proof Suppose that the conclusion is false. Then for any $n\in\mathbf{N}$ and any $E = \{w_1,\ldots,w_n\}\in \mathrm{F}_n(W)$, there exist Z-space $(X,\{F_{A.B}\})$, $A = \{x_1,\ldots,x_n\}\in \mathrm{F}_n(X)$ and $s\in \mathrm{C}(X,Y)$, such that

$$s(F_{A.A_I})\bigcap(\bigcap_{w\in E_I} H(w)) = \emptyset \quad \text{for any } I\in \mathrm{F}([n]),$$

or

$$s(F_{A.A_I}) \subset \bigcup_{w\in E_I}(Y\backslash H(w)) \quad \text{for any } I\in \mathrm{F}([n]). \tag{6}$$

Define mapping $G : W \to Y$ by $G(w) = Y\backslash H(w)$, $w\in W$. From (6) G is a Z-KKM mapping. By (i) $G(w)$ is compactly closed for any $w\in W$ and by (ii) $G(w_0)$ is compact. Follows Theorem 2.1

$$Y\backslash \bigcup_{w\in W} H(w) = \bigcap_{w\in W} G(w) \neq \emptyset.$$

This contradits (iii).

Theorem 2.4 *Let Y be a topological space, W a nonempty set. Suppose that mapping $T : W \to 2^Y$ satisfies*

(i) for each $y\in Y$, $T(y)\neq\emptyset$,

(ii) for each $w\in W$, $T^{-1}(w)$ is compactly open,

(iii) there exists a $w_0\in W$, such that $Y\backslash T^{-1}(w_0)$ is compact.

Then there exist $n\in\mathbf{N}$ and $E = \{w_1,\ldots,w_n\}\in \mathrm{F}_n(W)$, such that for each Z-space $(X,\{F_{A.B}\})$, each $A = \{x_1,\ldots,x_n\}\in \mathrm{F}_n(X)$ and each $s\in \mathrm{C}(X,Y)$, there exist $I\in \mathrm{F}([n])$ and $y\in Y$, such that $E_I\subset T(y)$ and $y\in s(F_{A.A_I})$.

Proof Suppose that the conclusion is false. Then for any $n\in\mathbf{N}$ and any $E = \{w_1,\ldots,w_n\}\in$

$\mathrm{F}_n(W)$, there exist Z-space $(X,\{F_{A.B}\})$, $A=\{x_1,\ldots,x_n\}\in \mathrm{F}_n(x)$ and $s\in \mathrm{C}(X,Y)$, such that for any $I\in \mathrm{F}([n])$,

$$s(F_{A.A_I})\bigcap(\bigcap_{w\in E_I} T^{-1}(w))=\emptyset,$$

or

$$s(F_{A.A_I})\subset \bigcup_{w\in E_I}(Y\backslash T^{-1}(w)).$$

Let $G:W\to 2^Y$ with $G(w)=Y\backslash T^{-1}(w)$. Then G is a Z-KKM mapping. By (ii) and (iii) $G(w)$ is compactly closed and $G(w_0)$ is compact. It follows from Theorem 2.1 that $\cap_{w\in W}G(w)\neq\emptyset$. So there exists $y_0\in Y$ such that

$$y_0\in\bigcap_{w\in W}G(w)=Y\backslash\bigcup_{w\in W}T^{-1}(w).$$

Hence $y_0\notin T^{-1}(W)=Y$, which is a contradiction.

Remark 2.2 In Theorem 2.4, if $Y=W=(X,\{F_{A.B}\})$ is a Z-space, and for each $x\in X$, $T(x)$ is Z-convex. Then Theorem 2.4 implies that there sxist $A\in \mathrm{F}_n(X)$, $I\in \mathrm{F}([n])$ and $x_0\in X$ such that $A_I\subset T(x_0)$ and $x_0\in F_{A.A_I}$. Since $T(x_0)$ is Z-convex $x_0\in F_{A.A_I}\subset T(x_0)$. That is x_0 is a fixed point of T. We extend the Browder's fixed point Theorem.

Theorem 2.5 *Let $(X,\{F_{A.B}\})$ be a Z-space. Suppose that $G:X\to 2^X$ satisfies*

(i) for each $x\in X$, $G(x)$ is compactly closed,

(ii) for each $x\in X$, $x\in G(x)$,

(iii) for each finite subset A of X, $G(\mathrm{co}(A))=G(A)$,

(iv) there exists a $x_0\in X$, $G(x_0)$ is compact.

Then $\cap_{x\in X}G(x)\neq\emptyset$.

Proof Suppose that $\cap_{x\in X}G(x)=\emptyset$. Let $T:X\to 2^X$, for $x\in X$, $T(x)=X\backslash G^{-1}(x)$. For any $x\in X$, we have: (1). there exists a $u\in X$ such that $x\notin G(u)$. So $u\in X\backslash G^{-1}(x)=T(x)$, hence $T(x)\neq\emptyset$; (2). since

$$u\in T^{-1}(x)\Leftrightarrow x\in T(u)=X\backslash G^{-1}(u)\Leftrightarrow x\notin G^{-1}(u)\Leftrightarrow u\in X\backslash G(x)$$

by (i), $T^{-1}(x)$ is compactly open; (3). for any finite subset A of $T(x)$,

$$\begin{aligned} A\subset X\backslash G^{-1}(x)&\Leftrightarrow A\cap G^{-1}(x)=\emptyset\\ &\Leftrightarrow x\notin G(A)=G(\mathrm{co}(A))\quad(\text{by (iii)})\\ &\Leftrightarrow \mathrm{co}(A)\cap G^{-1}(x)=\emptyset\Leftrightarrow \mathrm{co}(A)\subset X\backslash G^{-1}(x)=T(x).\end{aligned}$$

So $T(x)$ is Z-convex. Finally since

$$x\in X\backslash T^{-1}(x_0)\Leftrightarrow x_0\notin T(x)=X\backslash G^{-1}(x)\Leftrightarrow x_0\in G^{-1}(x)\Leftrightarrow x\in G(x_0)$$

by (iv), $X\backslash T^{-1}(x_0)$ is compact. It follows from Theorem 2.4 and Remark 2.2, there exists $u\in T(u)=X\backslash G^{-1}(u)$. Hence $u\notin G(u)$, this contradicts (ii).

Remark 2.3 Theorem 2.5 is a new version of KKM-theorem.

3. Some spacial cases

Definition 3.1 *Let X be a nonempty set, $(Y, \{F_{A.B}\})$ a Z-space and mapping $G : X \to 2^Y$. If for any $n \in \mathbf{N}$, any $E = \{x_1, \dots, x_n\} \in \mathrm{F}_n(X)$, there exists $A = \{y_1, \dots, y_n\} \in \mathrm{F}_n(Y)$ such that $F_{A.A_I} \subset G(E_I)$ for all $I \in \mathrm{F}([n])$. Then G is said a Z1-KKM mapping.*

Definition 3.2 *Let $(X, \{F_{A.B}\})$ be a Z-space, Y a topological space and mapping $G : D(\subset X) \to 2^Y$. If for any $n \in \mathbf{N}$ and any $A = \{x_1, \dots, x_n\} \in \mathrm{F}_n(D)$, there exist $B = \{u_1, \dots, u_n\} \in \mathrm{F}_n(X)$ and $s \in \mathrm{C}(X, Y)$ such that*

$$s(F_{B.B_I}) \subset G(A_I) \quad \text{for all } I \in \mathrm{F}([n]).$$

Then G is said a Z2-KKM mapping.

From Theorem 2.1 one have the following two theorems immediately.

Theorem 3.1 *Let X be a nonempty set, $(Y, \{F_{A.B}\})$ a Z-space. Suppose that $G : X \to 2^Y$ is a Z1-KKM mapping and for each $x \in X$, $G(x)$ is a compactly open (compactly closed) subset of Y. Then the family $\{G(x) : x \in X\}$ has the finite intersection property.*

Theorem 3.2 *Let $(X, \{F_{A.B}\})$ be a Z-space, Y a topological space. Suppose that $G : X \to 2^Y$ is a Z2-KKM mapping and for each $x \in X$, $G(x)$ is a compactly open (compactly closed) subset of Y. Then the family $\{G(x) : x \in X\}$ has the finite intersection property.*

From Theorem 2.3 we have

Theorem 3.3 *Let $(X, \{F_{A.B}\})$ be a Z-space, Y a topological space. Suppose that $H : D(\subset X) \to 2^Y$ satisfying*

(i) for any $x \in D$, $H(x)$ is compactly open,

(ii) $H(D) = Y$,

(iii) there exists a $x_0 \in D$ with $Y \backslash H(x_0)$ is compact.

Then there exist $n \in \mathbf{N}$ and $B = \{u_1, \dots, u_n\} \in \mathrm{F}_n(D)$, such that for any $A = \{x_1, \dots, x_n\} \in \mathrm{F}_n(X)$ and any $s \in \mathrm{C}(X, Y)$, there exist $I \in \mathrm{F}([n])$ and $x_0 \in F_{A.A_I}$ such that $s(x_0) \in \cap_{x \in B_I} H(x)$.

Theorem 3.4 *Let $(X, \{F_{A.B}\})$ be a Z-space, Y a topological space, E, F be nonempty subsets of $X \times Y$. Suppose*

(i) mapping $G : X \to 2^Y$, $G(x) = \{y \in Y : (x, y) \notin E\}$, is not Z2-KKM and

(ii) for any $y \in Y$, $\{x \in X : (x, y) \in F\}$ is Z-convex relative to $\{x \in X : (x, y) \in E\}$.

Then there exist $n \in \mathbf{N}$ and $A = \{x_1, \dots, x_n\} \in \mathrm{F}_n(X)$ such that for every $s \in \mathrm{C}(X, Y)$ there exist $I \in \mathrm{F}([n])$ and $x_0 \in F_{A.A_I}$ with $(x_0, s(x_0)) \in F$.

Proof From (i) there exist $n \in \mathbf{N}$ and $A = \{x_1, \dots, x_n\}$ such that for any $s \in \mathrm{C}(X, Y)$, there exists $I \in \mathrm{F}([n])$ such that $s(F_{A.A_I}) \not\subset G(A_I)$. So there exists a $x_0 \in F_{A.A_I}$ with $s(x_0) \notin G(x), \forall x \in A_I$, or $(x, s(x_0)) \in E, \forall x \in A_I$. So $A_I \subset \{x \in X : (x, s(x_0)) \in E\}$. By (ii)

$$F_{A.A_I} \subset \{x \in X : (x, s(x_0)) \in F\}.$$

Hence $(x_0, s(x_0)) \in F$.

Theorem 3.5 *Let $(X, \{F_{A,B}\})$ be a Z-space, Y a topological space and E a nonempty subset of $X \times Y$. Suppose that*

(i) mapping $G : X \to 2^Y$, $G(x) = \{y \in Y : (x, y) \notin E\}$ is Z2-KKM, and

(ii) for any $x \in X$, $\{y \in Y : (x, y) \in E\}$ is a compact oped subset of Y, and

(iii) there exists a $x_0 \in X$, such that $\{y \in Y : (x_0, y) \notin E\}$ is compact.

Then there exists a $y_0 \in Y$, such that $\{x \in X : (x, y_0) \in E\} = \emptyset$.

Proof This result follows from Theorem 3.2.

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KKM 定理的一个推广

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摘 要

本文引入了 Z- 空间概念，定义了一类新的映射：Z-KKM 映射，推广了著名的 FKKM 定理及其它各种形式的 KKM 定理.