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θ-Continuous Functions and θ-Compactness in Fuzzifying Topology *

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Abstract In this paper. the concepts of θ-continuous functions and θ-compactness in fyzzifying topology characterized in terms of θ-open sets is given. some properties of θ-continuous functions and θ-compactness are discussed.

Keywords θ-open sets. θ-continuous functions. θ-compactness.

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1. θ-open sets and θ-closed sets

Definition 1.1 *For any $A \subseteq X$, the fuzzifying θ-closure* $\mathrm{Cl}_\theta A$ *of A is defined as follows:*

$$x \in \mathrm{Cl}_\theta A := (\forall B)(B \in N_x \to \neg(A \cap \overline{B}) \equiv \emptyset)).$$

Theorem 1.1 *For any $x.A$,*

(1) $\models x \in \overline{A} \to x \in \mathrm{Cl}_\theta A$; *(2)* $\models (A \subseteq B) \to (\mathrm{Cl}_\theta A \subseteq \mathrm{Cl}_\theta B)$. *(3)* $\mathrm{Cl}_\theta(A \cup B) \equiv \mathrm{Cl}_\theta A \cup \mathrm{Cl}_\theta B$.

Proof Straightforward.

Definition 1.2 *Let Σ be a class of fuzzifying topological spaces. A unary fuzzy predicate $\mathcal{F}_\theta(\mathcal{T}_\theta) \in \mathcal{F}(P(X))$, called fuzzy θ-closed (θ-open), is given as follows:*

$$A \in \mathcal{F}_\theta := A \equiv \mathrm{Cl}_\theta A (A \in \mathcal{T}_\theta := A^C \in \mathcal{F}_\theta),$$

i.e.,

$$\mathcal{F}_\theta = \int_{P(X)} \inf_{x \in X \setminus A} (1 - \mathrm{Cl}_\theta A(x))/A. \; (\mathcal{T}_\theta = \int_{P(X)} \mathcal{F}_\theta(A^C)/A).$$

For any $A \subseteq X$, the fuzzy set of θ-interior points of A is called the θ-interior of A, and given as follows:

$$x \in \mathrm{Int}_\theta A := \neg(x \in \mathrm{Cl}_\theta(A^C)),$$

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i.e.,

$$\mathrm{Int}_\theta A = \int_X 1 - \mathrm{Cl}_\theta A^C(x)/x.$$

Theorem 1.2 (1) $\models A \equiv \mathrm{Int}_\theta A \leftrightarrow A \in \mathcal{T}_\theta$; (2) $\models A \in \mathcal{T}_\theta \to A \in \mathcal{T}$; (3) $\models A \in \mathcal{F}_\theta \to A \in \mathcal{F}$.

Proof (1) $[A \in \mathcal{T}_\theta] = [A^C \supseteq \mathrm{Cl}_\theta A^C] = \inf_{x \in (A^C)^C}(1 - \mathrm{Cl}_\theta A^C(x)) = [A \equiv \mathrm{Int}_\theta A]$.

(2) $[A \in \mathcal{T}_\theta] \le \inf_{x \in A}(1 - \overline{A^C}(x)) = \inf_{x \in A} N_x(A) = [A \in \mathcal{T}]$.

(3) Straightforward.

If $(X, \mathcal{T})$ is a fuzzifying topological space, then we can prove that $\mathcal{T}_\theta \in \mathcal{F}(P(X))$ is a fuzzifying topology on X, and called the θ-fuzzifying topology with respect to $\mathcal{T}$.

Definition 1.3 *Let $x \in X$. The θ-neighborhood system of x is denoted by $N_x^\theta \in \mathcal{F}(P(X))$ and defined as $A \in N_x^\theta := \exists B((B \in N_x) \mathbin{\&} (\overline{B} \subseteq A))$. Where $[a \mathbin{\&} b] = \max(0, [a] + [b] - 1)$.*

Theorem 1.3 *For any $A \in P(X), \models A \in \mathcal{T}_\theta \leftrightarrow \forall x(x \in A \to A \in N_x^\theta)$.*

Proof $[A \in \mathcal{T}_\theta] = \inf_{x \in A}(1 - \mathrm{Cl}_\theta A^C(x)) = \inf_{x \in A}(1 - \inf_B \min(1, 1 - N_x(B) + \sup_{y \in A^C} \overline{B}(y))) = \inf_{x \in A} \sup_B \max(0, N_x(B) - \inf_{y \in A^C}(1 - \overline{B}(y)) - 1) = \inf_{x \in A} \sup_B (N_x(B) \otimes \inf_{y \in A^C}(1 - \overline{B}(y))) = [(\forall x)(x \in A \to (\exists B)((B \in N_x) \mathbin{\&} (\overline{B} \subseteq A)))] = [\forall x(x \in A \to A \in N_x^\theta)]$.

Theorem 1.4 $\mathrm{Cl}_\theta A = \int_X 1 - N_x^\theta(A^C)/x$.

Proof $\mathrm{Cl}_\theta A(x) = \inf_B \min(1, 1 - N_x(B) + \sup_{y \in A} \overline{B}(y)) = 1 - \sup_B \max(0, N_x(B) - \sup_{y \in A} \overline{B}(y)) = 1 - \sup_B \max(0, N_x(B) + \inf_{y \in A}(1 - \overline{B}(y)) - 1) = 1 - [(\exists B)((B \in N_x) \mathbin{\&} (\overline{B} \subseteq A^C))] = 1 - N_x^\theta(A^C)$.

Theorem 1.5 *For any $A \subseteq X, \models A \in \mathcal{T} \to (\mathrm{Cl}_\theta A \equiv \overline{A})$.*

Proof First we prove $\models A \in \mathcal{T} \to (\neg(A \cap \overline{B} \equiv \emptyset) \to \neg(A \cap B) \equiv \emptyset)$. In fact,

$$\begin{aligned}
\mathcal{T}(A) \otimes [\neg(A \cap \overline{B}) \equiv \emptyset] &= \max(0, \inf_{x \in A} N_x(A) + \sup_{y \in A} \overline{B}(y) - 1) \\
&= \max(0, \sup_{y \in A} \overline{B}(y) - \sup_{x \in A}(1 - N_x(A))) \\
&\le \max(0, \sup_{x \in A}(\overline{B}(x) - 1 + N_x(A))) \\
&= \max(0, \sup_{x \in A}((B \cup B')(x) - 1 + N_x(A))) \\
&= \max(0, \sup_{x \in A}(\max B(x), B'(x)) - 1 + N_x(A))). \quad (*)
\end{aligned}$$

If $x \in B$, then $(*) = \max(0, \sup_{x \in A}(B(x) - 1 + N_x(A)) \le \sup_{x \in A} B(x)$.

If $x \notin B$, then

$$(*) = \max(0, \sup_{x \in A}(B'(x) - 1 + N_x(A)))$$

$$
\begin{aligned}
&= \max(0, \sup_{x\in A}(\inf_{D} \min(1, 1-N_x(A)+\sup_{y\in D}(B-\{x\}(y))-1+N_x(A))) \\
&\le \max(0, \sup_{x\in A}(1-N_x(A)+\sup_{y\in A}(B-\{x\}(y)-1+N_x(A))) \\
&= \sup_{y\in A}(B-\{x\})(y) = \sup_{y\in A} B(y) = [\neg(A\cap B\equiv\emptyset)].
\end{aligned}
$$

So, $[\neg(A\cap\overline{B}\equiv\emptyset)]\propto[\neg(A\cap B\equiv\Phi)]\ge\mathcal{T}(A)$. Therefore,

$$
\begin{aligned}
[\mathrm{Cl}_\theta A\equiv\overline{A}] &= \inf_{x\in X}(\mathrm{Cl}_\theta A(x)\propto\overline{A}(x)) \\
&= \inf(\inf_{B}(N_x(B)\propto\sup_{y\in A}\overline{B}(y))\propto\inf_{C}(N_x(c)\propto\sup_{y\in A}B(y))) \\
&\ge \inf_{x\in X}\inf_{B}((N_x(B)\propto\sup_{y\in A}\overline{B}(y))\propto(N_x(B)\propto\sup_{y\in A}B(y))) \\
&\ge \inf_{x}\inf_{B}(\sup_{y\in A}\overline{B}(y)\propto\sup_{y\in A}B(y))\ge\inf_{x\in X}\inf_{B}\mathcal{T}(A)=\mathcal{T}(A).
\end{aligned}
$$

2. θ-continuous functions

Definition 2.1 *Let $(X,\mathcal{T}),(Y,\mathcal{U})$ be two fuzzifying topological spaces. A unary fuzzy predicate $C_{S\theta}(C_\theta)\in\mathcal{F}(Y^X)$, called fuzzy strong θ-continuity, (fuzzy θ-continuity) is given as follows:*

$$
\begin{aligned}
C_{S\theta}(f) &:= (\forall X)(\forall u)(u\in N^Y_{f(x)}\to(\exists V)((V\in N^X_x)\mathbin{\&}(f(\overline{V})\subseteq u))), \\
C_{\theta}(f) &:= (\forall X)(\forall u)(u\in N^Y_{f(x)}\to(\exists V)((V\in N^X_x)\mathbin{\&}(f(\overline{V})\subseteq \bar{u}))).
\end{aligned}
$$

Theorem 2.1 *For any $f\in Y^X$, we set*
(1) $\alpha_1(f):=(\forall A)(f(\mathrm{Cl}_\theta A)\subseteq\overline{f(A)})$;
(2) $\alpha_2(f):=(\forall B)(\mathrm{Cl}_\theta f^{-1}(B)\subseteq f^{-1}(\overline{B}))$;
(3) $\alpha_3(f):=(\forall B)((B\in\mathcal{F}_Y)\to(f^{-1}(B)\in\mathcal{F}^X_\theta))$;
(4) $\alpha_4(f):=(\forall u)((u\in\mathcal{U})\to(f^{-1}(u)\in\mathcal{T}^X_\theta)$.
Then, $\models C_{S\theta}(f)\leftrightarrow\alpha_i(f), i=1,2,3,4$.

Proof (a) First, we prove $[C_{S\theta}(f)]\le[\alpha_1(f)]$. Since

$$[\alpha_1(f)]=\inf_{A}\inf_{y\in Y}\min(1,1-f(\mathrm{Cl}_\theta A)(y)+\overline{f(A)}(y)).$$

It is sufficient to show that for any A and $y\in Y$

$$\min(1,1-f(\mathrm{Cl}_\theta A)(y)+\overline{f(A)}(y))\ge[C_{S\theta}(f)].$$

If $f(\mathrm{Cl}_\theta A)(y)\le\overline{f(A)}(y)$, it holds obviously. Now assume $f(\mathrm{Cl}_\theta A)(y)>\overline{f(A)}(y)$. In fact, $[\neg(\overline{B}\cap A\equiv\emptyset]\le[\neg(f(\overline{B})\cap f(A)\equiv\emptyset]$. Therefore,

$$f(\mathrm{Cl}_\theta A)(y)-\overline{f(A)}(y)=\sup_{f(x)=y}\inf_{B}\min(1,1-N_x(B)+\sup_{z\in A}\overline{B}(z))-(1-N_y((f(A))^C))$$

$$
\begin{aligned}
&\leq \sup_x(\inf_B \min(1, 1 - N_x(B) + \sup_{z\in f(A)} f(\overline{B})(z)) - 1 + N_{f(x)}((f(A))^C)\\
&= \sup_x(N_{f(x)}((f(A))^C) - \sup_B \max(0, N_x(B) - \sup_{z\in f(A)} f(\overline{B})(z)))\\
&= \sup_x(N_{f(x)}((f(A))^C - \sup_B \max(0, N_x(B) + \inf_{z\in f(A)}(1 - f(\overline{B})(z)) - 1))\\
&= \sup_x(N_{f(x)}((f(A))^C) - \sup_B \max(0, N_x(B) + [f(\overline{B}) \subseteq ((f(A))^C) - 1))\\
&\leq \sup_x \sup_u(N_{f(x)}(u) - \sup_B \max(0, N_x(B) + [f(\overline{B}) \subseteq u] - 1)),
\end{aligned}
$$

$$
\begin{aligned}
&\min(1, 1 - f(\mathrm{Cl}_\theta A)(y) + \overline{f(A)}(y))\\
&\quad\geq \inf_x \inf_u \min(1, 1 - N_{f(x)}(u) + \sup_B \max(0, N_x(B) + [f(\overline{B}) \subseteq u] - 1)) = [C_{S\theta}(f)].
\end{aligned}
$$

(b) To prove $[\alpha_1(f)] \leq [\alpha_2(f)]$. For any $B \in P(Y)$, with $f^{-1}f(\mathrm{Cl}_\theta f^{-1}(B)) \supseteq \mathrm{Cl}_\theta f^{-1}(B), ff^{-1}(B) \subseteq B, \overline{ff^{-1}(B)} \subseteq \overline{B}, f^{-1}(\overline{ff^{-1}(B)}) \subseteq f^{-1}(\overline{B})$, we have

$$
\begin{aligned}
&[\mathrm{Cl}_\theta f^{-1}(B) \subseteq f^{-1}(\overline{B})] \geq [f^{-1}f(\mathrm{Cl}_\theta f^{-1}(B)) \subseteq f^{-1}(\overline{B})]\\
&\quad\geq [f^{-1}f(\mathrm{Cl}_\theta f^{-1}(B)) \subseteq f^{-1}(\overline{ff^{-1}(B)})] \geq [f(\mathrm{Cl}_\theta f^{-1}(B)) \subseteq \overline{ff^{-1}(B)}].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
[\alpha_2(f)] &= \inf_B[\mathrm{Cl}_\theta f^{-1}(B) \subseteq f^{-1}(\overline{B})] \geq \inf_B[f(\mathrm{Cl}_\theta f^{-1}(B)) \subseteq \overline{ff^{-1}(B)}]\\
&\geq \inf_A[f(\mathrm{Cl}_\theta A) \subseteq \overline{f(A)}] = [\alpha_1(f)].
\end{aligned}
$$

(c) We want to show that $[\alpha_2(f)] \leq [\alpha_3(f)]$. In fact for any $B \in P(Y)$,

$$
\begin{aligned}
&[\mathrm{Cl}_\theta f^{-1}(B) \subseteq f^{-1}(\overline{B})] \otimes \mathcal{F}_Y(B) = \inf_x \min(1, 1 - \mathrm{Cl}_\theta f^{-1}(B)(x) + f^{-1}(\overline{B})(x)) \otimes \mathcal{F}_Y(B)\\
&\quad= \inf_x \min(\mathcal{F}_Y(B), \max(0, \mathcal{F}_Y(B) + f^{-1}(\overline{B})(x) - \mathrm{Cl}_\theta f^{-1}(B)(x))),\\
&\mathcal{F}_Y(B) + f^{-1}(\overline{B})(x) - 1 = [\overline{B} \subseteq B] + f^{-1}(\overline{B})(x) - 1\\
&\quad\leq [f^{-1}(\overline{B}) \subseteq f^{-1}(B)] + f^{-1}(\overline{B})(x) - 1\\
&\quad= \inf_x \min(1, 1 - f^{-1}(\overline{B})(x) + f^{-1}(B)(x)) + f^{-1}(\overline{B})(x) - 1\\
&\quad\leq \min(f^{-1}(\overline{B})(x), f^{-1}(B)(x)) \leq f^{-1}(B)(x),\\
&[\mathrm{Cl}_\theta f^{-1}(B) \subseteq f^{-1}(\overline{B})] \otimes \mathcal{F}_Y(B) \leq \inf_x \min(1, 1 - \mathrm{Cl}_\theta f^{-1}(B)(x) + f^{-1}(B)(x))\\
&\quad= [f^{-1}(B) \in \mathcal{T}_\theta],\\
&[\mathrm{Cl}_\theta f^{-1}(B) \subseteq f^{-1}(B)] \leq (\mathcal{F}_Y(\overline{B}) \propto [f^{-1}(B) \in \mathcal{F}_\theta]).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
[\alpha_2(f)] &= [(\forall B)(\mathrm{Cl}_\theta f^{-1}(B) \subseteq f^{-1}(\overline{B})]\\
&\leq [(\forall B)((B \in \mathcal{F}_Y) \to (f^{-1}(B) \in \mathcal{F}_\theta^X))] = [\alpha_3(f)].
\end{aligned}
$$

(d)

$$\begin{aligned}[\alpha_3(f)] &= \inf_B \min(1, 1 - \mathcal{F}_Y(B) + \mathcal{F}_\theta^X(f^{-1}(B))) \\ &= \inf_B \min(1, 1 - \mathcal{U}(Y \backslash B) + \mathcal{T}_\theta(X \backslash f^{-1}(B))) \\ &= \inf_B \min(1, 1 - \mathcal{U}(Y \backslash B) + \mathcal{T}_\theta(f^{-1}(Y \backslash B))) \\ &= \inf_u \min(1, 1 - \mathcal{U}(u) + \mathcal{T}_\theta(f^{-1}(u))) = [\alpha_4(f)].\end{aligned}$$

(e) Finally, since $[C_{S\theta}(f)] = \inf_x \inf_u \min(1, 1 - N_{f(x)}^Y(u) + \sup_V \max(0, N_x(V) + [f(\overline{V}) \subseteq u] - 1))$, it suffices to show that for any $x \in X$ and $u \in P(Y)$,

$$\min(1, 1 - N_{f(x)}^X(u) + \sup_V \max(0, N_x(V) + [f(\overline{V}) \subseteq u] - 1)) \geq [\alpha_4(f)].$$

If $N_{f(x)}^Y(u) \leq \sup_Y \max(0, N_x(V) + [f(\overline{V}) \subseteq u] - 1)$, it holds obviously. Now assume $N_{f(x)}^Y(u) > \sup_V \max(0, N_x(V) + [f(\overline{V}) \subseteq u] - 1)$. In fact, $\overline{V} \subseteq f^{-1}(u)$ implies $f(\overline{V}) \subseteq u$. Therefore

$$\begin{aligned}&N_{f(x)}^Y(u) - \sup_V \max(0, N_x(V) + [f(\overline{V}) \subseteq u] - 1) \\ &\leq N_{f(x)}^Y(u) - \sup_V \max(0, N_x(V) + [\overline{V} \subseteq f^{-1}(u)] - 1 \\ &= N_{f(x)}^Y(u) - \sup_V \max(0, N_x(V) + \inf_{z \in (f^{-1}(V))^C} (1 - \overline{V}(z)) - 1) \\ &= \sup_{f(x) \in A \subseteq u} \mathcal{U}(A) - \sup_V \max(0, N_x(V) - \sup_{z \in f^{-1}(u^C)} \overline{V}(z)) \\ &\leq \sup_{f(x) \in A \subseteq u} (\mathcal{U}(A) - \sup_V \max(0, N_x(V) - \sup_{z \in f^{-1}(A^C)} \overline{V}(z))) \\ &= \sup_{f(x) \in A \subseteq u} (\mathcal{U}(A) - (1 - \inf_V \min(1, 1 - N_x(V) + \sup_{z \in f^{-1}(A^C)} \overline{V}(z)))) \\ &= \sup_{f(x) \in A \subseteq u} (\mathcal{U}(A) - (1 - \mathrm{Cl}_\theta f^{-1}(A^C)(x))) \\ &\leq \sup_{f(x) \in A \subseteq u} (\mathcal{U}(A) - \inf_{x \in f^{-1}(A)} (1 - \mathrm{Cl}_\theta f^{-1}(A^C)(x))) \\ &= \sup_{f(x \in A \subseteq u} (\mathcal{U}(A) - \mathcal{T}_\theta(f^{-1}(A))).\end{aligned}$$

$$\begin{aligned}&\min(1, 1 - N_{f(x)}(u) + \sup_V \max(0, N_x(V) + [f(\overline{V}) \subseteq u] - 1)) \\ &\geq \inf_{f(x) \in A \subseteq u} \min(1, 1 - \mathcal{U}(A) + \mathcal{T}_\theta(f^{-1}(A))) \\ &\geq \inf_B \min(1, 1 - \mathcal{U}(B) + \mathcal{T}_\theta(f^{-1}(B))) = [\alpha_4(f)].\end{aligned}$$

Theorem 2.2 *For any $f \in Y^X$, we set*

(1) $\beta_1(f) := (\forall A)(f(\mathrm{Cl}_\theta A) \subseteq \mathrm{Cl}_\theta f(A))$;

(2) $\beta_2(f) := (\forall B)(\mathrm{Cl}_\theta f^{-1}(B) \subseteq f^{-1}(\mathrm{Cl}_\theta B))$;
(3) $\beta_3(f) := (\forall B)(B \in \mathcal{F}_\theta^Y) \to f^{-1}(B) \in \mathcal{F}_\theta^X)$;
(4) $\beta_4(f) := (\forall B)(B \in \mathcal{U}_\theta \to f^{-1}(B) \in \mathcal{T}_\theta)$;
(5) $\beta_5(f) := (\forall B)(B \in \mathcal{U}) \to (\mathrm{Cl}_\theta f^{-1}(B) \subseteq f^{-1}(\overline{B})))$.
Then $\models C_\theta(f) \to \beta_i(f), i = 1,2,3,4,5$.

Proof The proof is similar to that of Theorem 2.1.

Theorem 2.3 *For any* $f \in Y^X, C(f) \models C_\theta(f)$.

Proof This proof is trivial.

3. θ-Compactness in fuzzifying topology

Definition 3.1 *Let Σ be a class of fuzzifying topological space. A unary fuzzy predicate $\Gamma_\theta \in \mathcal{F}(\Sigma)$, called fuzzy θ-compactness, is given as follows:*

$$\Gamma_\theta(X,\mathcal{T}) := (\forall \mathcal{A})(K_\theta(\mathcal{A},X) \to (\exists \mathcal{D})((\mathcal{D} \le \mathcal{A}) \mathbin{\&} K(\mathcal{D},X) \mathbin{\&} FF(\mathcal{D})))$$

where $K_\theta(\mathcal{A},X) := K(\mathcal{A},X) \mathbin{\&} (\mathcal{A} \subseteq \mathcal{T}_\theta), K(\mathcal{A},X) := (\forall x)(\exists A)(x \in A \in \mathcal{A})$.

Theorem 3.1 $\models \Gamma(X,\mathcal{T}) \to \Gamma_\theta(X,\mathcal{T})$.

Theorem 3.2 *Let $(X,\mathcal{T})$ be a fuzzifying topological space, and let*

$$\gamma_1 := (\forall \mathcal{A})((\mathcal{A} \subseteq \mathcal{T}_\theta) \mathbin{\&} fI(\mathcal{A}) \to (\exists x)(\forall A)((A \in \mathcal{A}) \to (x \in A)))$$
$$\gamma_2 := (\forall \mathcal{A})(\exists B)(((\mathcal{A} \subseteq \mathcal{F}_\theta) \wedge (B \in \mathcal{T}_\theta)) \mathbin{\&} (\forall \mathcal{D})((\mathcal{D} \le \mathcal{A}) \mathbin{\&} FF(\mathcal{D}) \to \neg(\cap\mathcal{D} \subseteq B)) \to \neg(\cap\mathcal{A} \subseteq B))$$

then $\models \Gamma_\theta(X,\mathcal{T}) \leftrightarrow \gamma_i, i = 1,2$.

Where, $fI(\mathcal{A}) := (\forall \mathcal{B})((\mathcal{B} \le \mathcal{A}) \mathbin{\&} FF(\mathcal{B}) \to (\exists x)(\forall B)((B \in \mathcal{B}) \to (x \in B)))$ and $\cap\mathcal{D} \in \mathcal{F}(X), x \in \cap\mathcal{D} := (\forall A)(A \in \mathcal{D} \to x \in A)$.

Proof (a) First, we prove $[\Gamma_\theta(X,\mathcal{T})] = [\gamma_1]$. For any $\mathcal{A} \in \mathcal{T}(P(X))$, we set

$$\mathcal{A}^C = \int_{A \in P(X)} \mathcal{A}(A)/A^C.$$

Then

$$\models (\mathcal{A} \subseteq \mathcal{T}_\theta) \leftrightarrow (\mathcal{A}^C \subseteq \mathcal{F}_\theta); \models FF(\mathcal{A}) \leftrightarrow FF(\mathcal{A}^C); \models (\mathcal{B} \le \mathcal{A}^C) \leftrightarrow (\mathcal{B}^C \le \mathcal{A});$$
$$\models K(\mathcal{A},X) \leftrightarrow \neg(\exists x)(\forall A)((A \in \mathcal{A}^C) \to (x \in A));$$
$$\models \neg fI(\mathcal{A}^C) \leftrightarrow (\exists \mathcal{B})((\mathcal{B} \le \mathcal{A}^C) \mathbin{\&} FF(\mathcal{B}) \mathbin{\&} \neg(\exists x)(\forall B)((B \in \mathcal{B}) \to (x \in B))) \leftrightarrow$$
$$(\exists \mathcal{B})((\mathcal{B}^C \le \mathcal{A}) \wedge FF(\mathcal{B}^C) \mathbin{\&} K(\mathcal{B}^C,X) \leftrightarrow (\exists \mathcal{D}((\mathcal{D} \le A) \wedge FF(\mathcal{D}) \mathbin{\&} K(\mathcal{D},X)$$
$$\models \gamma_1 \leftrightarrow (\forall \mathcal{A})((\mathcal{A}^C \subseteq \mathcal{F}_\theta) \mathbin{\&} fI(\mathcal{A}^C) \to (\exists x)(\forall A)((A \in \mathcal{A}^C) \to (x \in A))) \leftrightarrow$$
$$(\forall \mathcal{A})((\mathcal{A}^C \subseteq \mathcal{F}_\theta) \to (\neg(\exists x)(\forall A)((A \in \mathcal{A}^C) \to (x \in A)) \to \neg fI(\mathcal{A}^C))) \leftrightarrow$$
$$(\forall \mathcal{A})((\mathcal{A} \subseteq \mathcal{T}_\theta) \to (K(\mathcal{A},X) \to (\exists \mathcal{D})((\mathcal{D} \le \mathcal{A}) \mathbin{\&} K(\mathcal{D},X) \mathbin{\&} FF(\mathcal{D})))) \leftrightarrow$$
$$(\forall \mathcal{A})((\mathcal{A} \subseteq \mathcal{T}_\theta) \mathbin{\&} K(\mathcal{A},X) \to (\exists \mathcal{D})((\mathcal{D} \le \mathcal{A}) \mathbin{\&} K(\mathcal{D},X) \mathbin{\&} FF(\mathcal{D}))) \leftrightarrow \Gamma_\theta(X,\mathcal{T}).$$

(b) Let $B^C \in P(X)$ and for any $\mathcal{A} \in \mathcal{F}(P(X))$, then

$$
\begin{aligned}
&[(\mathcal{A} \cup \{B^C\}) \subseteq \mathcal{F}_\theta] \\
&\quad = \inf_A \min(1, 1 - (\mathcal{A} \cup \{B^C\}(A) + \mathcal{F}_\theta(A)) \\
&\quad = \inf_A \min(1, 1 - \mathcal{A}(A) + \mathcal{F}_\theta)) \wedge \inf_A \min(1, 1 - [A \in \{B^C\}] + \mathcal{F}_\theta(A)) \\
&\quad = [(\mathcal{A} \subseteq \mathcal{F}_\theta) \wedge (B^C \in \mathcal{F}_\theta)] = (\mathcal{A} \subseteq \mathcal{F}_\theta) \wedge (B \in \mathcal{T}_\theta)].
\end{aligned}
$$

For any $\mathcal{B} \in \mathcal{F}(P(X))$, let $\mathcal{D} = \mathcal{B} \backslash \{B^C\} \in \mathcal{F}(P(X))$.

$$
\mathcal{D}(A) = \begin{cases} \mathcal{B}(A), & A \neq B^C, \\ 0, & A = B^C. \end{cases}
$$

Then

$$
\mathcal{D} \le \mathcal{B}, \mathcal{D} \cup \{B^C\} \ge \mathcal{B}, [FF(\mathcal{D})] = [FF(\mathcal{B})], [\mathcal{D} \le \mathcal{A}] = [\mathcal{B} \le \mathcal{A} \cup \{B^C\})]
$$

and

$$
\begin{aligned}
&[(\forall \mathcal{D})((\mathcal{D} \le \mathcal{A}) \wedge FF(\mathcal{D}) \to (\exists x)(\forall A)(A \in \mathcal{D} \cup \{B^C\}) \to (x \in A)))] \\
&\quad = \inf_{\mathcal{D} \le \mathcal{A}} \min(1, 1 - [FF(\mathcal{D})] + \sup_x \inf_A (\mathcal{D} \cup \{B^C\})(A) \propto A(x)\} \\
&\quad \le \inf_{\mathcal{B} \le (\mathcal{A} \cup \{B^C\})} \min(1, 1 - [FF(\mathcal{B})] + \sup_x \inf_A \mathcal{B}(A) \propto A(x)) = [fI(\mathcal{A} \cup \{B^C\})].
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
&[\gamma_1] \otimes [((\mathcal{A} \subseteq \mathcal{F}_\theta) \mathbin{\&} (B \in \mathcal{T}_\theta)) \mathbin{\&} (\forall \mathcal{D})((\mathcal{D} \le \mathcal{A}) \mathbin{\&} FF(\mathcal{D}) \to \neg(\cap \mathcal{D}) \subseteq B))] \\
&\quad = [\gamma_1] \otimes [((\mathcal{A} \cup \{B^C\}) \subseteq \mathcal{F}_\theta) \mathbin{\&} (\forall \mathcal{D})((\mathcal{D} \le \mathcal{A}) \mathbin{\&} FF(\mathcal{D}) \to \\
&\qquad (\exists x)(\forall A)(A \in (\mathcal{D} \cup \{B^C\}) \to (x \in A)))] \\
&\quad = [\gamma_1] \otimes [((\mathcal{A} \cup \{B^C\}) \subseteq \mathcal{F}_\theta) \mathbin{\&} fI(\mathcal{A} \cup \{B^C\})] \\
&\quad \le [(\exists x)(\forall A)(A \in (\mathcal{A} \cup \{B^C\}) \to (x \in A))] \\
&\quad = [\neg(\cap \mathcal{A} \subseteq B)].
\end{aligned}
$$

Therefore, $[\gamma_1] \le \inf_{\mathcal{A}} \sup_B (((\mathcal{A} \subseteq \mathcal{F}_\theta) \mathbin{\&} (B \in \mathcal{F}_\theta)) \mathbin{\&} (\forall \mathcal{D})((\mathcal{D} \le \mathcal{A}) \mathbin{\&} FF(\mathcal{D}) \to \neg(\cap \mathcal{D} \subseteq B)) \to \neg(\cap \mathcal{A} \subseteq B)) = [\gamma_2]$.

Conversely,

$$
\begin{aligned}
&[\gamma_2] \otimes [(\mathcal{A} \subseteq \mathcal{F}_\theta) \mathbin{\&} fI(\mathcal{A})] \\
&\quad = [\gamma_2] \otimes [((\mathcal{A} \backslash \{B\}) \cup \{B\}) \subseteq \mathcal{F}_\theta) \mathbin{\&} fI((\mathcal{A} \backslash \{B\}) \cup \{B\})] \\
&\quad = [\gamma_2] \otimes [((\mathcal{A}' \subseteq \mathcal{F}_\theta) \wedge (B^C \in \mathcal{T}_\theta)) \mathbin{\&} ((\forall \mathcal{D})((\mathcal{D} \le \mathcal{A}') \mathbin{\&} FF(\mathcal{D}) \to \\
&\qquad (\exists x)(\forall A)(A \in (\mathcal{D} \cup \{B\}) \to (x \in A)))] \\
&\quad = [\gamma_2] \otimes [((\mathcal{A}' \subseteq \mathcal{F}_\theta) \wedge (B^C \in \mathcal{T}_\theta)) \mathbin{\&} ((\forall \mathcal{D})((\mathcal{D} \le \mathcal{A}') \mathbin{\&} FF(\mathcal{D}) \to \neg(\cap \mathcal{D} \subseteq B^C))] \\
&\quad \le [\neg(\cap \mathcal{A}' \subseteq B^C)] = [(\exists x)(\forall A)((A \in (\mathcal{A}' \cup \{B\})) \to (x \in A))] \\
&\quad = [(\exists x)(\forall A)((A \in \mathcal{A}) \to (x \in A))].
\end{aligned}
$$

Therefore,

$$[\gamma_2] \leq \inf_{\mathcal{A}}((\mathcal{A} \subseteq \mathcal{F}_\theta) \wedge fI(\mathcal{A}) \to (\exists x)(\forall A)((A \in \mathcal{A}) \to (x \in A))] = [\gamma_1].$$

Theorem 3.3 *For any fuzzifying topological space* $(X, \mathcal{T})$ *and* $(Y, \mathcal{U})$ *and* $f \in Y^X$ *is surjection,* $\models \Gamma_\theta(X, \mathcal{T}) \wedge C_\theta(f) \to \Gamma_\theta(Y, \mathcal{U})$.

Proof The proof is omitted.

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不分明化拓扑中的 θ- 连续函数和 θ- 紧性

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摘 要

本文在不分明化拓扑中从 θ- 开集出发给出了 θ- 连续函数和 θ- 紧性的概念，并讨论了它们的某些性质.