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Weak Hopf Algebras and Regular Monoids*

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Abstract: Weak Hopf algebra is defined. The aim of this work is to study the relations between the weak antipode of a weak Hopf algelra and the monoid of group-like elements. At first, the author shows some properties of weak Hopf algebras. And, for some weak Hopf algebras, some properties of the weak antipodes are given if the monoids of group-like elements are inverse or orthodox semigroups. At end, the author gets a weak Hopf algebra whose monoid of group-like elements is regular.

Keywords: weak Hopf algebra, weak autipode, group-like element, regular monoid.

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1. Introduction

In what follows, H always denotes a bialgebra over a field K.

It is known that for every bialgebra H the set of all group-like elements $G(H) = \{g \in H : \epsilon(g) = 1 \text{ and } \Delta(g) = g \otimes g\}$ forms a monoid with the multiplication of H.

Set $R = \mathrm{Hom}_K(H, H)$. For $f, g \in R$, define $f * g = m(f \times g)\Delta$. It is easy to see that R is an algebra over K with the multiplication $*$ and $u\epsilon$ is the identity of this algebra R.

A bialgebra H is called *Hopf algebra*[1] if there exists $S \in R$ such that $S * \mathrm{Id} = \mathrm{Id} * S = u\epsilon$. S is called an *antipode*. For a Hopf algebra H, the monoid $G(H)$ is a group.[1]

A bialgebra H is called a *left* (resp. *right*) *Hopf algebra*[2] if there is $S \in R = \mathrm{Hom}_K(H, H)$ such that $S * \mathrm{Id} = u\epsilon$ (resp., $\mathrm{Id} * S = u\epsilon$). S is called a *left* (resp. *right*) *antipode* of H.

Now, a bialgebra H is defined as a *weak Hopf algebra* if there is $T \in \mathrm{Hom}_K(H, H)$ such that $\mathrm{Id} * T * \mathrm{Id} = \mathrm{Id}$ and $T = T * \mathrm{Id} * T$. This T is called a *weak antipode*.

It is obvious that every left (resp. right) Hopf algebra with a left (resp. right) antipode S is a weak Hopf algebra and S is a weak antipode. Hence we have the following relations:

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Hopf algebra

Left Hopf algebra　　　　Right Hopf algebra

Weak Hopf algebra

Bialgebra

The aim of this work is to study the relations between bialgebras and their monoids of group-like elements. In particular, we will discuss the weak antipode and the monoid of group-like elements of a weak Hopf algebra.

In this paper, the notations and concepts about Hopf algebras and semigroups can be found respectively in [1] and [3].

2 On weak antipodes

Fact 2.1 *A bialgebra H is a weak Hopf algebra if and only if there is $T' \in \mathrm{Hom}_K(H, H)$ such that* $\mathrm{Id} * T' * \mathrm{Id} = \mathrm{Id}$.

Proof "only if" is obvious.

"if": Let $T = T' * \mathrm{Id} * T'$. Thus, $\mathrm{Id} * T * \mathrm{Id} = \mathrm{Id} * (T' * \mathrm{Id} * T') * \mathrm{Id} = (\mathrm{Id} * T' * \mathrm{Id}) * T' * \mathrm{Id} = \mathrm{Id} * T' * \mathrm{Id} = \mathrm{Id}$; $T * \mathrm{Id} * T = (T' * \mathrm{Id} * T') * \mathrm{Id} * (T' * \mathrm{Id} * T') = T' * (\mathrm{Id} * T' * \mathrm{Id}) * T' * \mathrm{Id} * T' = T' * (\mathrm{Id} * T' * \mathrm{Id}) * T' = T' * \mathrm{Id} * T' = T$. Then T is a weak antipode. □

Every Hopf algebra possesses a unique antipode. However, usually, a weak Hopf algebra may have one or more weak antipodes; a left (right, resp.) Hopf algebra may also have one or more left (right, resp.) antipodes (see [2]).

Let $\tau: H \otimes H \to H \otimes H$ be the twist mapping defined by $\tau(h \otimes h') = h' \otimes h$ for all $h, h' \in H$ and let $m^{op}: H \otimes H \to H$, $\Delta^{op}: H \to H \otimes H$ be the mappings defined by $m^{op} = m\tau$, $\Delta^{op} = \tau\Delta$ (the opposite multiplication and the opposite comultiplication). Then it is esay to see[4] that $H_{00} = (H; m, u, \Delta, \epsilon)$, $H_{01} = (H; m, u, \Delta^{op}, \epsilon)$, $H_{10} = (H; m^{op}, u, \Delta, \epsilon)$, $H_{11} = (H; m^{op}; u, \Delta^{op}, \epsilon)$ are bialgebras as well (of course, H_{00} is the bialgebra we started with).

For a Hopf algebra H with an antipode S, we know in [4] that H_{11} is a Hopf algebra with the antipode S; and H_{01} (resp. H_{10}) is a Hopf algebra if and only if S is invertible, in this case S^{-1} is the antipode of H_{01} (resp. H_{10}). For weak Hopf algebras, we have the following statements:

Theorem 2.2 *Let $H = H_{00}$ be a weak Hopf algebra with a weak antipode T. Then*

(*a*) *the bialgebra H_{11} is a weak Hopf algebra with the weak antipode T;*

(*b*) *the bialgebra H_{01} (resp. H_{10}) is a weak Hopf algebra if and only if there exists $P \in \mathrm{Hom}_K(H, H)$ such that $m_{123}(\mathrm{Id} \otimes T \otimes \mathrm{Id})\Delta_2 = m_{321}(\mathrm{Id} \otimes P \otimes \mathrm{Id})\Delta_2$ where $\Delta_2 = (\Delta \otimes \mathrm{Id})\Delta$, $m_{123} = m(m \otimes \mathrm{Id})$ and $m_{321} = m\tau(m \otimes \mathrm{Id})(\tau \otimes \mathrm{Id})$. In this case, H_{01} (resp. H_{10}) has the weak antipode $T^q = P * \mathrm{Id} * P$;*

(*c*) *the bialgebra H_{01} (resp. H_{10}) is a weak Hopf algebra if one of the following conditions is satisfied:*

(*i*) *Let $E = \{e_j: j \in J\}$ be a basis of H. If for any $i, j, k \in J$, there is $a_k \in J$ such that $e_j a_k e_i = e_i (T e_k) e_j$;*

(ii) *As a semigroup about the multiplication* m, H *is 3-rewriteable for the permutation* $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$;

(iii) H *is commutative or cocommutative.*

Proof (a) follows directly from the definition of H_{11}. For (b), let us prove the statement concerning H_{10}. Similarly for H_{01}.

"if": For H_{10}, $\mathrm{Id} * P * \mathrm{Id} = \mathrm{Id} * (m^{op}(P \otimes \mathrm{Id})\Delta) = m\tau(\mathrm{Id} \otimes m\tau(P \otimes \mathrm{Id})\Delta)\Delta = m\tau(\mathrm{Id} \otimes m\tau)(\mathrm{Id} \otimes P \otimes \mathrm{Id})(\mathrm{Id} \otimes \Delta)\Delta = m\tau(m\tau \otimes \mathrm{Id})(\mathrm{Id} \otimes P \otimes \mathrm{Id})(\Delta \otimes \mathrm{Id})\Delta = m_{321}(\mathrm{Id} \otimes P \otimes \mathrm{Id})\Delta_2 = m_{123}(\mathrm{Id} \otimes T \otimes \mathrm{Id})\Delta_2$; and for H_{00}, $\mathrm{Id} = \mathrm{Id} * T * \mathrm{Id} = m(\mathrm{Id} \otimes m(T \otimes \mathrm{Id})\Delta)\Delta = m(\mathrm{Id} \otimes m)(\mathrm{Id} \otimes T \otimes \mathrm{Id})(\mathrm{Id} \otimes \Delta)\Delta = m_{123}(\mathrm{Id} \otimes T \otimes \mathrm{Id})\Delta_2$. Then for H_{10}, $\mathrm{Id} * P * \mathrm{Id} = \mathrm{Id}$. It means that H_{10} is a weak Hopf algebra with a weak antipode $T^q = P * \mathrm{Id} * P$ by Fact 2.1 and its proof.

"only if": Let P be a weak antipode of H_{10}. As prove above, for H_{10}, $\mathrm{Id} = \mathrm{Id} * P * \mathrm{Id} = m_{321}(\mathrm{Id} \otimes P \otimes \mathrm{Id})\Delta_2$; for H_{00}, $\mathrm{Id} = \mathrm{Id} * T * \mathrm{Id} = m_{123}(\mathrm{Id} \otimes T \otimes \mathrm{Id})\Delta_2$. Hence $m_{321}(\mathrm{Id} \otimes P \otimes \mathrm{Id})\Delta_2 = m_{123}(\mathrm{Id} \otimes T \otimes \mathrm{Id})\Delta)\Delta_2$.

(c) (i) Let $P \in \mathrm{Hom}_K(H, H)$ such that $P(e_k) = a_k$ for any $k \in J$. Then $e_j(Pe_k)e_i = e_i(Te_k)e_j$. It follows $z(Py)x = x(Ty)z$ for any $x, y, z \in H$. Thus, $m_{123}(\mathrm{Id} \otimes T \otimes \mathrm{Id})\Delta_2(a) = m_{321}(\mathrm{Id} \otimes P \otimes \mathrm{Id})\Delta_2(a)$ for any $a \in H$. By (b), H_{01} is a weak Hopf algebra.

(ii) By [5], a semigroup A is said to be n-rewriteable if for any sequence of elements $x_1, \cdots, x_n \in A$ we have $x_1x_2\cdots x_n = x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)}$ for some nontrivial permutation σ of n symbols. H is 3-rewriteable for the permutation $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$, hence for any $i, j, k \in J$, $e_j(Te_k)e_i = e_i(Te_k)e_j$. Then the result follows from (i).

(iii) H is 3-rewriteable for $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ if it is commutative. Then the result follows from (ii). If H is cocommutative, then $\Delta = \tau\Delta$ and for any $a \in H$, $m_{123}(\mathrm{Id} \otimes T \otimes \mathrm{Id})\Delta_2(a) = m_{123}(\mathrm{Id} \otimes T \otimes \mathrm{Id})(\tau\Delta \otimes \mathrm{Id})\tau\Delta(a) = m_{123}(\mathrm{Id} \otimes T \otimes \mathrm{Id})(\sum_{(a)} a_{(3)} \otimes a_{(2)} \otimes a_{(1)}) = \sum_{(a)} a_{(3)}T(a_{(2)})a_{(1)} = m_{321}(\mathrm{Id} \otimes T \otimes \mathrm{Id})\Delta_2(a)$. Thus, $m_{123}(\mathrm{Id} \otimes T \otimes \mathrm{Id})\Delta_2 = m_{321}(\mathrm{Id} \otimes T \otimes \mathrm{Id})\Delta_2$. Then the result follows from (b). □

It is interesting to compare this proposition with the similar result on Hopf algebras[4].

According to [6], a bialgebra H is called *almost cocommutative* if there exists an invertible element $Q \in H \otimes H$ such that for all $a \in H$, $\tau\Delta(a) = Q\Delta(a)Q^{-1}$. Moreover, a bialgebra H is called a *quasitriangular bialgebra* if H is almost cocommutative for an invertible element $Q \in H \otimes H$ such that $(\Delta \otimes \mathrm{Id})(Q) = Q^{13}Q^{23}$, $(\mathrm{Id} \otimes \Delta)(Q) = Q^{13}Q^{12}$ where $Q = \sum_i a_i \otimes b_i$, $Q^{12} = \sum_i a_i \otimes b_i \otimes 1$, $Q^{23} = \sum_i 1 \otimes a_i \otimes b_i$ and $Q^{13} = \sum_i a_i \otimes 1 \otimes b_i$ (see also [7]).

Proposition 2.3 *If* (H, Q) *is a quasitriangular weak Hopf algebra with a weak antipode* T, *then the triangular relation* $Q^{12}Q^{13}Q^{23} = Q^{23}Q^{13}Q^{12}$ *holds, as well as the relations hold:*

$$(m(\mathrm{Id} \otimes T)\Delta \otimes \mathrm{Id})(Q) = 1 \otimes 1, \tag{1}$$

$$(\epsilon \otimes \mathrm{Id})(Q) = 1 = (\mathrm{Id} \otimes \epsilon)(Q). \tag{2}$$

When H_{01} *is a weak Hopf algebra with weak antipode* T^q, *then the dual relation of* (1) *holds*: $(\mathrm{Id} \otimes m\tau(T^q \otimes \mathrm{Id})\Delta)(Q) = 1 \otimes 1$.

Proof (2) is proved in [7].

$$Q = \sum_i a_i \otimes b_i = \sum_i (\mathrm{Id})(a_i) \otimes b_i = \sum_i (\mathrm{Id} * T * \mathrm{Id})(a_i) \otimes b_i$$

$$= \sum_i m_{123}(\mathrm{Id} \otimes T \otimes \mathrm{Id})\Delta_2(a_i) \otimes b_i$$

$$= [m_{123}(\mathrm{Id} \otimes T \otimes \mathrm{Id})(\Delta \otimes \mathrm{Id}) \otimes \mathrm{Id}](\Delta \otimes \mathrm{Id})(Q)$$

$$= [m_{123}(\mathrm{Id} \otimes T \otimes \mathrm{Id})(\Delta \otimes \mathrm{Id}) \otimes \mathrm{Id}](Q^{13}Q^{23})$$

$$= (m_{123} \otimes \mathrm{Id})(\mathrm{Id} \otimes T \otimes \mathrm{Id} \otimes \mathrm{Id})(\Delta \otimes \mathrm{Id} \otimes \mathrm{Id})(\sum_i a_1 \otimes 1 \otimes b_i)(\sum_i 1 \otimes a_i \otimes b_i)$$

$$= \sum_{i,j} (m_{123} \otimes \mathrm{Id})(\sum_{(a_i)} a_{i1} \otimes T a_{i2} \otimes a_j \otimes b_i b_j) \quad (\mathrm{let}\, \Delta(a_i) = \sum_{(a_i)} a_{i1} \otimes a_{i2})$$

$$= \sum_{i,j} (\sum_{(a_i)} a_{i1}(T a_{i2}) a_j \otimes b_i b_j) = \sum_i (\sum_{(a_i)} a_{i1}(T a_{i2}) \otimes b_i) Q.$$

Hence $Q = [\sum_i (\sum_{(a_i)} a_{i1}(T a_{i2}) \otimes b_i)]Q$. Then $1 \otimes 1 = \sum_i (\sum_{(a_i)} a_{i1}(T a_{i2}) \otimes b_i)$. But

$$\sum_i (\sum_{(a_i)} a_{i1}(T a_{i2}) \otimes b_i) = \sum_i (m(\sum_{(a_i)} a_{i2} \otimes T a_{i2}) \otimes b_i)$$

$$= (m \otimes \mathrm{Id})(\sum_i (\sum_{(a_i)} a_{i1} \otimes T a_{i2}) \otimes b_i)$$

$$= (m \otimes \mathrm{Id})(\mathrm{Id} \otimes T \otimes \mathrm{Id})(\sum_i (\sum_{(a_i)} a_{i1} \otimes a_{i2}) \otimes b_i)$$

$$= (m \otimes \mathrm{Id})(\mathrm{Id} \otimes T \otimes \mathrm{Id})(\Delta \otimes \mathrm{Id})(Q) = (m(\mathrm{Id} \otimes T)\Delta \otimes \mathrm{Id})(Q).$$

Hence $[m(\mathrm{Id} \otimes T)(\Delta \otimes \mathrm{Id}](Q) = 1 \otimes 1$.

It is easy to see that $(H_{01}, \tau Q)$ is a quasitriangular weak Hopf algebra with the weak antipode T^q, then by (1), $(m(\mathrm{Id} \otimes T^q)\tau\Delta \otimes \mathrm{Id})(\tau Q) = 1 \otimes 1$. Then $(\mathrm{Id} \otimes m(\mathrm{Id} \otimes T^q)\tau\Delta)(Q) = 1 \otimes 1$. It follows that $(\mathrm{Id} \otimes m\tau(T^q \otimes \mathrm{Id})\Delta)(Q) = 1 \otimes 1$. □

Through the concept of a quasitrianglar weak Hopf algebra and Proposition 2.3, we can find some relations between weak Hopf algebras and the theory of quantum groups which is important in mathematical physics (see [7]).

3 On the monoid of group-like elements

Now we discuss the relations between the weak antipode of a weak Hopf algebra and the regularity of the monoid of group-like elements.

In a semigroup S, for every $a \in S$, denote $V(a)$ as the set of all inverses of a; $E(S)$ as the set of all idempotents of S; for every $e \in E(S)$, denote $E(e)$ as the set $V(e)$ (see [3]).

At first, suppose that $T(G(H)) \subset G(H)$. It is equivalent to $\Delta T|_{G(H)} = (T \otimes T)\Delta|_{G(H)}$.

Proposition 3.1 *For a weak Hopf algebra H with a weak antipode T, if $T(G(H)) \subset G(H)$, then $G(H)$ is a regular monoid.*

Proof For every $g \in G(H)$, $g = \mathrm{Id}(g) = (\mathrm{Id} * T * \mathrm{Id})(g) = gT(g)g$. But $T(g) \in G(H)$. Then

g is a regular element in $G(H)$. So $G(H)$ is regular. □

Proposition 3.2 *For a weak Hopf algebra H with a weak antipode T, if $T(G(H)) \subset G(H)$ and $G(H)$ is an inverse semigroup, then the following statements hold:*

(*a*) *For every $g \in G(H)$, $T(g) = g'$ the unique inverse of g in $G(H)$;*

(*b*) *For every other weak antipode T_1 (if it exists), $T_1|_{G(H)} = T|_{G(H)}$;*

(*c*) *T is a semigroup anti-morphism on $G(H)$, and is an algebra anti-morphism on $KG(H)$;*

(*d*) $\mathrm{Ker}(T|_{G(H)}) = \{(a, b) \in G(H) \times G(H): T(a) = T(b)\}$ *is a congruence on $G(H)$.*

Proof (a) $g = gT(g)g$ and $T(g) = T(g)gT(g)$. Then $T(g) \in V(g)$. But $|V(g)| = 1$ (see [3]). Hence $T(g) = g'$.

(b) For any $g \in G(H)$, $gT_1(g)g = g$ and $T_1(g)gT_1(g) = T_1(g)$. So $T_1(g) \in V(g)$. But $|V(g)| = 1$. Thus $T_1(g) = T(g) = g'$.

(c) By (a), $T(g) = g'$ for every $g \in G(H)$. By [3, p. 131, Prop. 1.4], $(g_1g_2)' = g_2'g_1'$ for $g_1, g_2 \in G(H)$. So T is a semigroup anti-morphism on $G(H)$, moreover, is an algebra anti-morphism on $KG(H)$.

(d) For any $a, b, c \in G(H)$, if $T(a) = T(b)$, then $T(ac) = T(c)T(a) = T(c)T(b) = T(bc)$. Similarly, $T(ca) = T(cb)$. By [3, p. 21], Ker $(T|_{G(H)})$ is a congruence. □

Theorem 3.3 *For a weak Hopf algebra H with a weak antipode T, if $T(G(H)) \subset G(H)$ and $G(H)$ is an orthodox semigroup, then the following statements hold:*

(*a*) *For any $a, b \in G(H)$, $T(ab)\, D\, T(b)T(a)$ and $D_{T(ab)} = D_{T(b)T(a)} = D_{ab}$;*

(*b*) *For every $g \in G(H)$, $V(g) = E(T(g)g)T(g)E(gT(g))$;*

(*c*) *For $a, b \in G(H)$, if $T(a) = T(b)$, then $V(a) = V(b)$;*

(*d*) *If $V(a) = V(b)$ implies $T(a) = T(b)$ for any $a, b \in G(H)$, then* $\mathrm{Ker}(T|_{G(H)}) = \{(a, b) \in G(H) \times G(H): T(a) = T(b)\}$ *is the smallest inverse semigroup congruence on $G(H)$.*

Proof (a) $T(a) \in V(a)$, $T(b) \in V(b)$, $T(ab) \in V(ab)$. By [3, p. 187], $T(b)T(a) \in V(ab)$. And by [3, p. 46], $T(ab) \in D_{ab}$, $T(b)T(a) \in D_{ab}$. Hence $T(ab)\, D\, T(b)T(a)$ and $D_{T(ab)} = D_{T(b)T(a)} = D_{ab}$.

(b) Since $T(g) \in V(g)$, the result follows from [3, p. 189, Prop. 1.9].

(c) Since $T(a) \in V(a)$, $T(b) \in V(b)$, then $V(a) \cap V(b) \neq \emptyset$ when $T(a) = T(b)$. By [3, p. 189, Theorem 1.10], $V(a) = V(b)$.

(d) By (c), we know that in this case for $a, b \in G(H)$, $T(a) = T(b)$ if and only if $V(a) = V(b)$. Hence $\mathrm{Ker}(T|_{G(H)}) = \{(a, b) \in G(H) \times G(H): V(a) = V(b)\}$. By [3, p. 190, Theorem 1.12], it is the smallest inverse semigroup congruence on $G(H)$. □

Usually, $T(G(H)) \subset G(H)$ does not hold. In this case, we want to know whether $G(H)$ is a regular monoid.

About $G(H)$ of a left Hopf algebra H, the authors[2] got that for a left Hopf algebra H which is pointed as a coalgebra, $G(H)$ is a group. Here a coalgebra is called *pointed* if its all simple subcoalgebras are 1-dimensional.

About $G(H)$ of a weak Hopf algebra H, we obtain the similar result:

Theorem 3.4 *Let H be a weak Hopf algebra which is pointed as a coalgebra, then $G(H)$ is a reg-*

ular monoid.

Proof The *coradical*[8] of a coalgebra is the sum of all simple subcoalgebra. Since H is pointed, the coradical H_0 of H is the direct sum of all 1-dimensional subcoalgebras Kg ($g \in G(H)$). Let $\{H_n\}$ be the coradical filtration[8] of H. Then $H_0 \subset H_1 \subset \cdots \subset H$ with $H = \cup_i H_i$ and $\Delta(H_n) \subset \sum_{i=0}^{n} H_i \otimes H_{n-i}$. By the splitting theorem of [9], there is a coalgebra projection π of H onto H_0 such that π is the identity on H_0.

Now we prove that for every $g \in G(H)$, $\pi(gHg) \subset gH_0g$ by inducation on the coradical filtration of H. Since $gH_0g \subset H_0$, π is the identity on gH_0g. Since

$$\Delta H_n \subset \sum_{i=0}^{n} H_i \otimes H_{n-i} \subset H_0 \otimes H + H \otimes H_{n-1} \text{ for every } n \geq 1,$$

one has

$$\Delta(gH_ng) = \Delta(g)\Delta(H_n)\Delta(g) \subset gH_0g \otimes gHg + gHg \otimes gH_{n-1}g.$$

Suppose $\pi(gH_{n-1}g) \subset gH_0g$. Thus

$$\begin{aligned}\Delta\pi(gH_ng) &= (\pi \otimes \pi)\Delta(gH_ng) \subset (\pi \otimes \pi)(gH_0g \otimes gHg + gHg \otimes gH_{n-1}g) \\ &= \pi(gH_0g) \otimes \pi(gHg) + \pi(gHg) \otimes \pi(gH_{n-1}g) \subset \\ &\quad gH_0g \otimes H_0 + H_0 \otimes gH_0g\end{aligned}$$

by induction. Let $H_0 = gH_0g \oplus X$ where gH_0g has a basis $\{gh_ig\}$ for some $h_i \in G(H)$. Then

$$\begin{aligned}\Delta\pi(gH_ng) &\subset gH_0g \otimes (gH_0g \oplus X) + (gH_0g \oplus X) \otimes gH_0g \\ &= gH_0g \otimes gH_0g + gH_0g \otimes X + X \otimes gH_0g.\end{aligned}$$

If $y \in \pi(gH_ng)$, then $y \in H_0$ by the above. Writing $y = \sum \alpha_i gh_ig + \sum \beta_i x_i$. Then

$$\Delta y = \sum \alpha_i gh_ig \otimes gh_ig + \sum \beta_j x_j \otimes x_j.$$

Since $y \in \pi(gH_ng)$, we have

$$\Delta y \in gH_0g \otimes gH_0g + gH_0g \otimes X + X \otimes gH_0g$$

by the above. But

$$x_j \otimes x_j \notin gH_0g \otimes gH_0g + gH_0g \otimes X + X \otimes gH_0g \text{ for any } x_j,$$

therefore $\beta_i = 0$ for any j. Thus

$$y = \sum \alpha_i gh_ig \in gH_0g.$$

It means $\pi(gH_ng) \subset gH_0g$ for any n. Hence $\pi(gHg) \subset gH_0g$. $g = \pi(g)$ since π is the identity on gH_0g. And $g = gT(g)g$. Then

$$g = \pi(gT(g)g) \in \pi(gHg) \subset gH_0g.$$

So we can write $g = \sum \gamma_i gh_ig$ for some $\gamma_i \in K$. Hence $g = gh_ig$ for some i, since the elements of $G(H)$ is linear independent over K by [1]. It says that any $g \in G(H)$ is regular. □

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弱 Hopf 代数与正则幺半群

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摘　要

本文定义了弱 Hopf 代数并研究了弱 Hopf 代数的弱对极与类群元幺半群的关系. 首先, 本文给出弱 Hopf 代数的一些基本性质; 然后, 对类群元幺半群是逆半群或纯正半群的某些弱 Hopf 代数, 描述了其弱对极的一些性质; 最后, 给出一类其类群元幺半群为正则半群的弱 Hopf 代数.