Journal of Mathematical Research & Exposition
Vol. 19, No. 2, 332-336, May 1999

Weak Global Dimension and Endomorphisms of Modules*

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Abstract: Using endomorphisms of modules in this paper, we give the characterizations for rings of weak global dimension $\leq n$ where $n \geq 0$. Let R be a ring, we partially answer the question: When has any finitely presented R-module M the nonfinitely resolution: $0 \to M \to F_0 \to F_1 \to \cdots \to F_n \to \cdots$, where each F_i is finitely generated projective, $i = 0, 1, 2, \cdots$?

Keywords: weak global dimension, flat module, absolutely pure dimension, endomorphism.

Classification: AMS(1991) 16E10, 16D60/CLC O153.3

Document code: A **Article ID**: 1000-341X(1999)02-0332-05

1. Introduction

Throughout this paper R is a ring with unity $1 \neq 0$, R-Mod (resp. mod-R) will denote the category of left (resp. right) unital R-modules, also $_RM$ (resp. M_R) indicate that M is in R-mod (resp. mod-R). Unless otherwise metioned, we will be working in R-mod. For all terminology, the reader is referred to [1].

We know that a ring R is a VN regular ring (weak global dimension is 0) if and only if every R-module is flat, and if and only if for any $\alpha \in R$ there exists an element $\beta \in R$ such that $\alpha = \alpha\beta\alpha$. The first result of the paper (Proposition 1) show that R is a VN regular ring if and only if the co-image $M/\ker\alpha$ of every flat left R-module M under an endomorphism α is again flat. Moreover, with the analegous methods of [2], we give characterizations for rings of weak global dimension $\leq$ n, where n is any natural number.

Following [3], let R be a commutative ring, then R is coherent if and only if for any R-module A, and for any projective resolution $P_{n+1} \to P_n \to \cdots \to P_0 \to A \to 0$ of A, with P_{n+1}, P_n finitely generated, we can find fintitely generated projective R-modules $P_{n+2}, P_{n+3}, \cdots$, such that $\cdots \to P_{n+3} \to P_{n+2} \to P_{n+1} \to P_n \to \cdots \to P_0 \to A \to 0$ is exact (see Proposition 2.2[3]). Therefore, let R be a commutative coherent ring, then for any finitely presented R-module M, M has the non-finitely resolution: $\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$, where P_i is finitely generated projective

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Naturally, there is the question: Let R be a ring, when has finitively presented R-module M the nonfinitely resolution: $0 \to M \to F_0 \to F_1 \to \cdots \to F_n \to \cdots$ where F_i is finitely generated projective, and $i = 0, 1, 2, \cdots$?. Another purpose of this paper is to partially answer the question above.

2 Main results

Proposition 1 *Let R be a ring, T denote the class of all flat left R-modules. Then the following are equivalent:*

(1) *R is a von Neumann regular ring (VN regular ring);*

(2) *If $M \in T$ and $f \in \mathrm{end}_R M$, then* co-ker$f = M/\mathrm{Im} f \in T$;

(3) *If M_R is flat and $f \in \mathrm{end} M$, then $M/\mathrm{Im} f$ is flat in mod-R.*

Proof Since the weak global dimension requires no left-right distinction, clearly it suffices to prove (1) $\Leftarrow\Rightarrow$ (2). An immediate result is (1) $=\Rightarrow$ (2).

(2) $=\Rightarrow$ (1). Let $M \in T$ and H is a submodule of M, we first show that $M/H \in T$. In fact, let $\sigma: F \to H$ be surjective with F free, obviously $F \in T$. Define $\Phi \in \mathrm{end}(M \oplus F)$ by $\Phi(x, y) = (\sigma y, 0)$, then $\mathrm{Im}\,\Phi = H \oplus 0$, so $(M/H) \oplus F \simeq (M \oplus F)/\mathrm{Im}\,\Phi$ is in T from e (2). Therefore M/H is flat, as required.

Now assume N any R-module. Notice again there is a surjective homomorphism $\epsilon: F_N \to N$, where F_N is a free R-module, so the following sequence is exact: $0 \to \ker\epsilon \to F_N \to F_N/\ker\epsilon \to 0$. By the forgoing discussion, $F_N/\ker\epsilon$ is flat. Then N is flat because $N \simeq F_N/\ker\epsilon$.

Theorem 2 *Let R be a ring, then the following statements are equivalent:*

(1) *The weak global dimension of R is ≤ 1;*

(2) *If $M \in T$ and $f \in \mathrm{end}_R M$, then* co-im $f = M/\ker f \in T$.

Proof (1) $=\Rightarrow$ (2) is clear.

(2) $=\Rightarrow$ (1). Let $M \in T$, L is a submodule of M. We need prove that L is in T. Consider $\sigma: F \to L$, where F is a free R-module and σ is a surjective homomorphism. Define $\partial \in \mathrm{end}(M \oplus F)$ by $\partial(m, y) = (\sigma y, 0)$. Then $\ker\partial = M \oplus \ker\sigma$, so $M \oplus F/\ker\sigma = M \oplus F/M \oplus \ker\sigma \simeq F/\ker\sigma \simeq L$. By (2), we have L is in T.

Remark Recall taht a ring R with weak global dimension ≤ 1 if and only if T is closed under taking submodules. But Theorem 2 above show that the weak global dimenson of R is ≤ 1 if and only if the co-image $M/\ker f$ of every flat left R-module M under an endomorphism f is again flat, thus weakening the usual requirement that T is closed under taking submodules.

Generally, for the rings of weak global dimension at most n, where $n \geq 0$ is any integer, we obtain the following theorem.

Theorem 3 *Let R be a ring, $F_n = \{{}_R M \mid \text{Flat } \dim M \leq n\}$, then the following ar equivalent:*

(1) *weak global* $\dim R \leq n$;

(2) *If* $M \in F_n$ *and* $f \in \mathrm{end}_R M$, *then* co-ker$f = M / \mathrm{Im} f \in F_n$.

Proof (1) =⇒ (2) trival.

(2) =⇒ (1) Assume $M \in F_n$, L is a submodule of M, we show that $M/L \in F_n$, In fact, there are following exact sequences:

$$0 \longrightarrow L \longrightarrow M \longrightarrow M/L \longrightarrow 0$$

and

$$F \xrightarrow{\sigma} L \longrightarrow 0,$$

where F is free. F is flat so $F \in F_n$. Define $\Phi \in \mathrm{end}(M \oplus F)$ by $\Phi(m, y) = (\sigma y, 0)$. Then $\mathrm{Im}\Phi = L \oplus 0$ so $M \oplus F / \mathrm{Im}\Phi = M \oplus F / L \oplus 0 \simeq M/L \oplus F$. By the shifting theorem of flat dimension, $M \oplus F \in F_n$, so from (2) we have $M/L \oplus F \in F_n$, consequently Flat. dim $(M/L \oplus F) \leq n$, as required.

Now let N be any R-module. Consider the exact sequence $0 \longrightarrow \ker\partial \rightarrow P \xrightarrow{\partial} N \rightarrow 0$ with P projective. By the known result above, $P/\ker\partial \in F_n$, so $N \in F_n$ because $N \simeq P/\ker\partial$.

Note that we have left-right symmetry in Theorem 3(2) above beacuse weak global dimension of R requires no left-right distinction.

As before, T denote the class of all flat left R-modules. If $f \in \mathrm{end}_R M$ with $M \in T$, and if $M/\mathrm{Ker} f$ is also in T, then the exact sequence

$$0 \rightarrow \ker f \rightarrow M \rightarrow M/\ker f \rightarrow 0$$

implies $0 \rightarrow (M/\ker f)^+ \rightarrow M^+ \rightarrow \ker f^+ \rightarrow 0$ is exact, where $(M/\ker f)^+$, M^+, $\ker f^+$ denote the character module of $M/\ker f$, M, $\ker f$ resp.. Consequently, $\ker f^+$ is a direct summand of M^+, so $\ker f \in T$. This suggests the question:

What's the characterization of the ring R for which $M \in T$ and $f \in \mathrm{end}_R M$ implies that $\ker f$ is in T?

The following theorem give an answer on the question above.

Theorem 4 *let* R *be a ring, then the following are equivalent*:

(1) *the weak global dimension of* R *is* ≤ 2;

(2) *If* $M \in T$ *and* $\partial \in \mathrm{end}_R M$, then $\ker\partial \in T$.

Proof (1) =⇒ (2) Given $M \in T$ and $\partial \in \mathrm{end}_R M$, then we have the following exact sequence:

$$0 \rightarrow \ker\epsilon \rightarrow M \rightarrow M/\mathrm{Im}\,\partial \rightarrow 0$$

By (1), Flat. dim$M/\mathrm{Im}\,\partial \leq 2$, but M is flat, so $\ker\partial$ is in T.

(2) =⇒ (1) Given ${}_R N$, obviously there is the exact sequence $0 \rightarrow L \rightarrow F \xrightarrow{\sigma} N \rightarrow 0$ with F' free. Similarly, there is the exact sequence $F' \xrightarrow{\delta} L \rightarrow 0$ with F free. Define $\Phi \in \mathrm{end}(F' \oplus F)$ by

$$\Phi(x, y) = (0, \delta x)$$

so $\ker\Phi = \ker\delta \oplus F$. Since $F' \oplus F \in T$, condition (2) implies that $\ker\Phi \in T$, so $\ker\delta \in T$. Therefore there is the exact sequence of N:

$$0 \rightarrow \ker\delta \rightarrow F' \rightarrow F \xrightarrow{\sigma} N \rightarrow 0$$

It shows that is Flat. dim$N \leq 2$. Since $_RN$ was arbitrary, this implies (1).

The arguments simular to those used above can be used to prove the following results:

Theorem 2' *Let R be a ring, $n \geq 1$. Then the following are equivalent:*

(1) *The weak global dimension of R is $\leq n+1$;*

(2) *If $M \in F_n$ and $f \in \text{end}_R M$, then $M/\ker f \in F_n$.*

Theorem 4' *Let R be a ring, $n \geq 1$. Then the following are equivalent:*

(1) *The weak global dimension of R is $\leq n+2$;*

(2) *If $M \in F_n$ and $f \in \text{end}_R M$, then $\ker f \in F_n$.*

For a commutative ring R, [3] defined the finitely presented dimension of R. It measures how far away a ring is from being Noetherian. Following [3], let R be a commutative ring, then R is coherent if and only if for any R-module A, and for any projective resolution $P_{n+1} \to P_n \to \cdots \to P_0 \to A \to 0$ of A, with P_{n+1}, P_n finitely generated, we can find fintitely generated projective R-modules $P_{n+2}, P_{n+3}, \cdots$, such that $\cdots \to P_{n+3} \to P_{n+2} \to P_{n+1} \to P_n \to \cdots \to P_0 \to A \to 0$ is exact (see Proposition 2.2[3]). Therefore let R be commutative coherent ring, then for any finitely presented R-module M, M has the nonfinitely resolution: $\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$, where P_i is finitely generated projective. Naturally, there is the question:

Let R be a ring, when has finitively presented R-module M the nonfinitely resolution:

$$0 \to M \to F_0 \to F_1 \to \cdots \to F_n \to \cdots$$

where F_i is finitely generated projective, and $i = 0, 1, 2, \cdots$?.

The next purpose of this paper is to discuss the question above.

Theorem 5 *Let R be a left and right coherent ring. If the absolutely pure dimension of R as right R-module (i.e. apd (R_R), see [4]) = 1, then for any finitely presented right R-module C, C^* has the following nonfinitely resolution.*

$$0 \to C^* \to F_0 \to F_1 \to \cdots \to F_n \to \cdots$$

where F_i if finitely generated projective, $i = 0, 1, 2, \cdots$.

Proof We firstly prove the following assertion. Let C be a finitely presented torsionless left R-module, then C has the resolution: $0 \to C \to F_0 \to B \to 0$, where F_0 is finitely generated free and B is also finitely presented torsionless. In fact, by [5], C^* is a finitely presented right R-module. Consider the exact sequence $F_1 \xrightarrow{f} F_0 \to C^* \to 0$, then we have the exact sequence $0 \to C^{**} \to F_0^* \xrightarrow{f^*} F_1^*$. Let $B = \text{Im} f^*$, so $0 \to C^{**} \to F_0^* \to B \to 0$ is exact, and B is also finitely presented torsionless left R-module. Using [6] Theorem 5, C is reflexive. So $0 \to C \to F_0^* \to B \to 0$ exacts, as required.

Now suppose C is a finitely presented right R-module. By [5], C^* is a finitely presented left R-module. Since C^* is torsionless, then it follows that $0 \to C^* \to F_0 \to B \to 0$ exacts, where B is also finitely presented torsionless, and F_0 is finitely generated free. Using the assertion to prove above again, then we have the following exact sequence: $0 \to B \to F_1 \to B_1 \to 0$, where F_1 is finitely gene rated free and B is also finitely presented torsionless. Therefore $0 \to C^* \to F_0 \to F_1 \to B_1 \to 0$ exacts. Consequently, there is the nonfinitely exact sequence: $0 \to C^* \to F_0 \to F_1 \to F_2 \to \cdots$, where F_i is finitely generated free and i= 0, 1, 2, ⋯.

Proposition 6 *Let R be a commutative coherent ring. If the absolutely pure dimension of R_R is 0, then for any finitely presented left R-module M, M has the following nonfinitely resolution*: $0 \to M \to F_0 \to F_1 \cdots \to F_n \to \cdots$ *where each F_i is finitely generated projective.*

Proof We simply note that any finitely presented left R-module M is torsionless from [7] Theorem 2.3. Since apd(R_R) = 0, $Ext_R^2(A, R) = 0$, for any finitely presented right R-module A. So by [6] Theorem 4, M is reflexive. Similar to the proof of Theorem 5 above, the result following immediately.

Acknowledgement The author would like to thank Professors Zhou Boxun and Tong wenting for their helps and encouragements.

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弱总体维数和模的自同态

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摘　要

本文用模的自同态, 给出弱总体维数 $\leq n$ 的环的特征, 其中 $n \geq 0$. 设 R 为环, 部分地回答了下列问题: 何时任意有限表现 R-模 M 有无穷分解: $0 \to M \to F_0 \to F_1 \to \cdots \to F_n \to \cdots$, 其中每个 F_i 均是有限生成投射的, $i = 1, 2, \cdots$?