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On a Class of Subspace Lattices*

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Abstract: Let $V_n(q)$ be the n-dimensional vector space over the finite field with q elements, K a k-dimensional subspace and $C[n,k]$ the set of the subspaces S such that $S \cap K \neq O$. We show that $C[n,k]$ is Sperner and unimodal and point out all maximum-sized antichains in $C[n,k]$. For the Whitney number W_m of $C[n,k]$, we show that $W_m^2 - qW_{m-1}W_{m+1}$ has nonnegative coefficients as a polynomial in q and that $W_0 \leq W_n \leq W_1 \leq W_{n-1} \leq W_2 \leq \cdots$.

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1. Introduction

Let $C(n,k)$ be the collection of all subsets of an n-set X which intersect a fixed k-subset Y of X. Then $C(n,k)$ is a natural generalization for the subset lattice. Lih [10] first observed this and showed that $C(n,k)$ is Sperner and unimodal. Griggs [7] further showed that $C(n,k)$ has several stronger properties, including the nested chain decomposition, the LYM inequality and the log concavity of Whitney numbers. He also determined all maximal-sized antichains in $C(n,k)$. In this paper we consider the analogous problem for finite vector spaces.

Let $V_n(q)$ denote the n-dimensional vector space (n-space, for short) over the finite field with q elements and $L_n(q)$ denote the lattice of subspaces of $V_n(q)$. Let K be a fixed k-subspace and $C[n,k]$ the set of the subspaces S such that $S \cap K \neq O$ where O denote the null space. $C[n,k]$ is an extension for the subspace lattice. In particular, $C[n,n]$ is isomorphic to $L_n(q)$ while $C[n,1]$ is isomorphic to $L_{n-1}(q)$.

The object of this paper is twofold. First, we show that $C[n,k]$ is Sperner and unimodal. The method of Lih used in [10] for $C(n,k)$ is still valid for $C[n,k]$. However, we will present a simple approach to show that $C[n,k]$ is both Sperner and unimodal. Secondly, we investigate the Whitney numbers W_m of $C[n,k]$, which can be regarded as an extension of the q

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-binomial coefficient. We present a few formulas about W_m to explore further properties of $C[n, k]$. For example, we show that $W_m^2 - qW_{m-1}W_{m+1}$ has nonnegative coefficients as a polynomial in q for all m which extends Butler's corresponding result about q-binomial coefficients. And hence in particular, $W_m(n, k)$ is log concave for m. We also show that $W_0 \le W_n \le W_1 \le W_{n-1} \le W_2 \le \cdots$, which is a necessary condition on the Whitney numbers such that there exists a nested chain decomposition in $C[n, k]$ similar to that in $C(n, k)$ obtained by the bracketing construction.

2 Terminology

In this paper we will employ the terminology and notation of Griggs[8] and Chen and Rota[5].

A finite poset (partially ordered set) P is ranked if there exists a function $r: P \longrightarrow \{0, 1, 2, \cdots\}$ such that $r(x) = 0$ for minimal elements $x \in P$ and $r(y) = r(z) + 1$ if y covers $z \in P$. $r(x)$ is the rank of x, and the rank $r(P)$ of P is the maximal rank of the elements of P. Let P_m denote the mth rank set consisting of elements of rank m in P. The Whitney number $W_m = |P_m|$ is called the mth Whitney number (of the second kind) of P. P is Sperner if $\max_m W_m = \max\{|A| : A$ is an antichain in $P\}$, the common value is called the Sperner number of P. P is unimodal if the sequence of Whitney numbers of P is unimodal, i.e., $W_0 \le \cdots \le W_{l-1} \le W_l \ge W_{l+1} \ge \cdots \ge W_n$ for some l, where $r(p) = n$. Similarly, P is log concave if the sequence of Whitney numbers of P is log concave: $W_m^2 \ge W_{m-1}W_{m+1}$ for all m. It is clear that P is log concave implies that P is unimodal.

Let $G = G(X, Y; E)$ be a finite bipartite graph with sets of vertices X and Y and with a set of edges E between X and Y. We will denote G for short by $X \cup Y$. A (complete) matching of X into Y in G is a subset of E which meets every member of X exactly once and every member of Y at most once. Denote $X \quad Y$, provided that there exists a matching of X into Y.

Suppose that in a ranked poset P there exists a matching between every consecutive pair of rank sets from the smaller to the larger one, then P is said to have the matching property. In particular, if there exist matchings

$$P_0 \quad P_1 \quad \cdots \quad P_l \quad P_{l+1} \cdots \quad P_n,$$

then P is said to have the unimodal matching property.

Throughout this paper we let K be a fixed k-subspace of $V_n(q)$ and $C[n, k]$ the set of subspaces S of $V_n(q)$ such that $S \cap K \ne O$. For convenience we adjoin the null space O to $C[n, k]$, thus the extended set (still denote by $C[n, k]$) turns out to be a sublattice of $L_n(q)$ and the ranks agree with the dimensions of the spaces. Let $\left[{n \atop m}\right]$ denote the q-binomial coefficient, i.e., the number of m-subspaces of $V_n(q)$. It is well known that

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ n-m \end{bmatrix} = \frac{(q^n - 1)(q^n - q)\cdots(q^n - q^{m-1})}{(q^m - 1)(q^m - q)\cdots(q^m - q^{m-1})}.$$

3 Sperner and Unimodal Properties

In this section, we show that $C[n, k]$ is Sperner and unimodal. We will employ the following lemma (see, e. g. , Stanley[13]) to obtain this result.

Lemma 1 *If* P *has the unimodal matching property*

$$P_0 \quad P_1 \quad \cdots \quad P_l \quad P_{l+1} \cdots \quad P_n,$$

then P *is rank unimodal and Sperner with the Sperner number* W_l.

Thus it suffices to show that there exists a matching in every bipartite graph consisting of the consecutive pair of rank sets. The most popular characterization for the existence of a matching in a bipartite graph is Hall's marriage theorem, but the following lemma sometimes may be more convenient (see, e. g. , Canfield[4]). As usual, $\deg(x)$ denotes the degree of a vertex x in the graph.

Lemma 2 *For a bipartite graph* $X \cup Y$, *if* $\deg(x) > 0$ *for any* $x \in X$ *and* $\min_{x \in X} \deg(x) \geq \max_{y \in Y} \deg(y)$, then $X \quad Y$.

Let C_m be the m th rank set of $C[n, k]$. For $S \in C_m$, let S^* (resp. S_*) denote the collection of subspaces in $C[n, k]$ which cover (resp. , are covered by) S. Note that $|S^*|$ (resp. $|S_*|$) is the degree of the vertex S in the bipartite graph $C_m \cup C_{m+1}$ (resp. $C_m \cup C_{m-1}$).

Lemma 3 *For* $S \in C_m$, $|S^*| = \begin{bmatrix} n-m \\ 1 \end{bmatrix}$, *and* $|S_*| = \begin{bmatrix} m \\ 1 \end{bmatrix}$ *if* $\dim(S \cap K) \geq 2$ *or* $|S_*| = \begin{bmatrix} m-1 \\ 1 \end{bmatrix}$ *if* $\dim(S \cap K) = 1$.

Proof Suppose that T covers $S \in C_m$ in $L_n(q)$. Then $T \cap K \neq O$, which implies that $T \in C_{m+1}$, hence $|S^*| = \begin{bmatrix} n-m \\ 1 \end{bmatrix}$.

Now suppose that T is covered by $S \in C_m$ in $L_n(q)$. If $\dim(S \cap K) \geq 2$ then $T \cap K \neq O$, hence $T \in C_{m-1}$, so $|S_*| = \begin{bmatrix} m \\ 1 \end{bmatrix}$; if $\dim(S \cap K) = 1$, then there are just $\begin{bmatrix} m-1 \\ 1 \end{bmatrix}$ $(m-1)$-subspaces in S containing $S \cap K$ from the self duality of $L_n(q)$, so $|S_*| = \begin{bmatrix} m-1 \\ 1 \end{bmatrix}$. [ ]

Theorem 1 $C[n, k]$ *has the unimodal matching property and hence is both unimodal and Sperner, with the Sperner number* W_N *where* N *is the least positive integer* $\geq \frac{1}{2}n$.

Proof If $m < N$ then $n-m \geq m + 1$. Hence in the bipartite graph $C_m \cup C_{m+1}$,

$$\min \deg(S) = \begin{bmatrix} n-m \\ 1 \end{bmatrix} \geq \begin{bmatrix} m+1 \\ 1 \end{bmatrix} \geq \max \deg(T)$$

for arbitrary $S \in C_m$ and $T \in C_{m+1}$ from Lemma 3. Thus $C_m \quad C_{m+1}$ from Lemma 2. Similarly, if $m > N$ then $C_m \quad C_{m-1}$. Consequently the theorem follows immediately from Lemma 1. [ ]

The method of Griggs[7] for finding all maximal-sized antichains of $C(n, k)$ can carry over almost without change to $C[n, k]$, hence we only list the related results by omitting the proof.

Theorem 2 *The only maximum-sized antichains in* $C[n, k]$ *are the largest rank set(s)*

(i) C_N, and

(ii) C_{N-1} if $n = 2N-1$ and $k \geq N+1$, and

(iii) C_{N+1} if $n = 2N$ and $k = 1$.

Remark Griggs' methods used to show that $C(n, k)$ is nested, LYM and log concave are not adapted to $C[n, k]$. For example, he induced a nested chain decomposition of $C(n, k)$ from the symmetric chain decomposition of the subset lattice obtained by bracketing construction, but a similar chain decomposition of the subspace lattice has not been found so far.

4 Whitney Numbers

In the present section, we focus our attention on the Whitney numbers $W_m(n, k)$ of $C[n, k]$. $W_m(n, k)$ can be regarded as an extension of the q-binomial coefficients and shares many of the properties of the q-binomial coefficient. In particular, $W_m(n, n) = \begin{bmatrix} n \\ m \end{bmatrix}$ and $W_m(n, 1) = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}$.

In what follows, we always set $\begin{bmatrix} n \\ m \end{bmatrix} = 0$ and $W_m(n, k) = 0$ unless $0 \leq m \leq n$.

To formulate the Whitney numbers, we need the following enumeration result.

Lemma 4 *For arbitrary $0 \leq i \leq k$, the number of m-subspaces S of $V_n(q)$ such that* $\dim(S \cap K) = i$ *is* $q^{(k-i)(m-i)} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} n-k \\ m-i \end{bmatrix}$.

Proof Note that for each $0 \leq i \leq k$, the number of ordered bases of m-subspaces S such that $\dim(S \cap K) = i$ is $(q^n - q^k)(q^n - q^{k+1}) \cdots (q^n - q^{k+m-i-1})$ (see, e.g., Goldman and Rota [6]). But the number of ordered bases of this kind of m-subspace is $(q^m - q^i)(q^m - q^{i+1}) \cdots (q^m - q^{m-1})$. The quotient of these two quantities gives the number of m-subspaces S of $V_n(q)$ such that $\dim(S \cap K) = i$:

$$\frac{(q^n - q^k)(q^n - q^{k+1}) \cdots (q^n - q^{k+m-i-1})}{(q^m - q^i)(q^m - q^{i+1}) \cdots (q^m - q^{m-1})} = q^{(k-i)(m-i)} \begin{bmatrix} n-k \\ m-i \end{bmatrix}. \qquad []$$

Thus we obtain

$$W_m(n, k) = \begin{bmatrix} n \\ m \end{bmatrix} - q^{km} \begin{bmatrix} n-k \\ m \end{bmatrix} \tag{1}$$

$$= \sum_{i=1}^{k} q^{(k-i)(m-i)} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} n-k \\ m-i \end{bmatrix}. \tag{2}$$

Chen and Rota [5] gave the formula

$$q^{km} \begin{bmatrix} n-k \\ m \end{bmatrix} = \sum_{i=0}^{k} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} n-i \\ m-i \end{bmatrix},$$

hence

$$W_m(n, k) = \sum_{i=1}^{k} (-1)^{i-1} q^{\binom{i}{2}} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} n-i \\ m-i \end{bmatrix}.$$

Fomulas (1), (2) and (3) are the extension of the corresponding results for the q-binomial coefficients, respectively. We can also extend the q-Pascal triangles about q-binomial coefficients

$$\begin{bmatrix} n+1 \\ m \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix} + q^{n-m+1}\begin{bmatrix} n \\ m-1 \end{bmatrix} \tag{4}$$

and

$$\begin{bmatrix} n+1 \\ m \end{bmatrix} = \begin{bmatrix} n \\ m-1 \end{bmatrix} + q^{m}\begin{bmatrix} n \\ m \end{bmatrix} \tag{5}$$

as follows

Lemma 5 *Let $1 \leqslant m \leqslant n$ and $1 \leq k \leq n$. Then*

$$W_m(n+1,k) = W_m(n,k) + q^{n-m+1}W_{m-1}(n,k) \tag{6}$$

and

$$W_m(n+1,k) = \begin{bmatrix} n \\ m-1 \end{bmatrix} + q^{m}W_m(n,k-1). \tag{7}$$

Proof From the recursion (4) it follows that

$$\begin{aligned} W_m(n+1,k) &= \begin{bmatrix} n+1 \\ m \end{bmatrix} - q^{km}\begin{bmatrix} n+1-k \\ m \end{bmatrix} \\ &= \left\{\begin{bmatrix} n \\ m \end{bmatrix} + q^{n-m+1}\begin{bmatrix} n \\ m-1 \end{bmatrix}\right\} - q^{km}\left\{\begin{bmatrix} n-k \\ m \end{bmatrix} + q^{n-k-m+1}\begin{bmatrix} n-k \\ m-1 \end{bmatrix}\right\} \\ &= \left\{\begin{bmatrix} n \\ m \end{bmatrix} - q^{km}\begin{bmatrix} n-k \\ m \end{bmatrix}\right\} + q^{n-m+1}\left\{\begin{bmatrix} n \\ m-1 \end{bmatrix} - q^{k(m-1)}\begin{bmatrix} n-k \\ m-1 \end{bmatrix}\right\} \\ &= W_m(n,k) + q^{n-m+1}W_{m-1}(n,k). \end{aligned}$$

Similarly, it immediately follows from the recursion (5) that

$$\begin{aligned} W_m(n+1,k) &= \begin{bmatrix} n+1 \\ m \end{bmatrix} - q^{km}\begin{bmatrix} n+1-k \\ m \end{bmatrix} \\ &= \left\{\begin{bmatrix} n \\ m-1 \end{bmatrix} + q^{m}\begin{bmatrix} n \\ m \end{bmatrix}\right\} - q^{km}\begin{bmatrix} n+1-k \\ m \end{bmatrix} \\ &= \begin{bmatrix} n \\ m-1 \end{bmatrix} + q^{m}\left\{\begin{bmatrix} n \\ m \end{bmatrix} - q^{(k-1)m}\begin{bmatrix} n+1-k \\ m \end{bmatrix}\right\} \\ &= \begin{bmatrix} n \\ m-1 \end{bmatrix} + q^{m}W_m(n,k-1). \end{aligned}$$

The proof is then completed. [ ]

Butler[3] showed that for $m \leq l$

$$\begin{bmatrix} n \\ m \end{bmatrix}\begin{bmatrix} n \\ l \end{bmatrix} - q^{l-m+1}\begin{bmatrix} n \\ m-1 \end{bmatrix}\begin{bmatrix} n \\ l+1 \end{bmatrix}$$

has nonnegative coefficients as a polynomial in q. We can generalize this by the recursion (6).

Lemma 6 *Given $k \geq 1$. Then for all $n \geq k$ and $m \leq l$,*

$$W_m(n,k)W_l(n,k) - q^{l-m+1}W_{m-1}(n,k)W_{l+1}(n,k)$$

has nonnegative coefficients as a polynomial in q.

Proof The lemma is true for $n = k$ from Butler's result described above since $W_m(n, n) = \begin{bmatrix} n \\ m \end{bmatrix}$. Now we proceed by induction on n. Assume that the lemma is true for n and consider the case $n + 1$. For $m \leq l$, it follows from the recursion (6) that

$$\begin{aligned} & W_m(n+1,k)W_l(n+1,k) - q^{l-m+1}W_{m-1}(n+1,k)W_{l+1}(n+1,k) \\ = & \left\{W_m(n,k) + q^{n-m+1}W_{m-1}(n,k)\right\}\left\{W_l(n,k) + q^{n-l+1}W_{l-1}(n,k)\right\} \\ & -q^{l-m+1}\left\{W_{m-1}(n,k) + q^{n-m+2}W_{m-2}(n,k)\right\}\left\{W_{l+1}(n,k) + q^{n-l}W_l(n,k)\right\} \\ = & \left\{W_m(n,k)W_l(n,k) - q^{l-m+1}W_{m-1}(n,k)W_{l+1}(n,k)\right\} \\ & + q^{2n-m-l+2}\left\{W_{m-1}(n,k)W_{l-1}(n,k) - q^{l-m+1}W_{m-2}(n,k)W_l(n,k)\right\} \\ & + q^{n-m+1}\left\{W_{m-1}(n,k)W_l(n,k) - q^{l-m+2}W_{m-2}(n,k)W_{l+1}(n,k)\right\} \\ & + q^{n-l+1}\left\{W_m(n,k)W_{l-1}(n,k) - q^{l-m}W_{m-1}(n,k)W_l(n,k)\right\} \end{aligned}$$

has nonnegative coefficients since each of the four terms in the sum has nonnegative coefficients by the inductive hypothesis (the last term is zero if $m = l$). This completes the proof.
[ ]

A special case of the above lemma gives one of the main results of this paper.

Theorem 3 *$W_m^2(n,k) - qW_{m-1}(n,k)W_{m+1}(n,k)$ has nonnegative coefficients as a polynomial in q.*

In particular, $W_m(n, k)$ is log concave for m, hence we have

Corollary *$C[n,k]$ is log concave.*

To state the final result in this section, we need another expression for the Whitney numbers

$$\begin{aligned} W_m(n,k) &= \begin{bmatrix} n \\ m \end{bmatrix} - q^{km}\begin{bmatrix} n-k \\ m \end{bmatrix} \\ &= \sum_{i=1}^{k} q^{(i-1)m}\left\{\begin{bmatrix} n-i+1 \\ m \end{bmatrix} - q^m\begin{bmatrix} n-i \\ m \end{bmatrix}\right\} \\ &= \sum_{i=1}^{k} q^{(i-1)m}\begin{bmatrix} n-i \\ m-1 \end{bmatrix}. \end{aligned} \qquad (8)$$

We also require

Lemma 7 *For $s \leq r$ and $t \leq \frac{1}{2}r$,*

$$f(r,s,t) = \begin{bmatrix} r-s \\ t \end{bmatrix} - q^{s(r-2t)}\begin{bmatrix} r-s \\ r-t \end{bmatrix} \geq 0.$$

Proof The case $r = 2t$ is trivial and hence let $t < \frac{1}{2} r$.

If $t > r-s$ then $r-t > r-s$, hence $f(r, s, t) = 0$; if $t < s$ then $f(r, s, t) = \begin{bmatrix} r-s \\ t \end{bmatrix} \geq 0$; if $s \leq t \leq r-s$, then by the formula $\begin{bmatrix} r-s \\ t \end{bmatrix} \begin{bmatrix} t \\ s \end{bmatrix} = \begin{bmatrix} r-s \\ r-t \end{bmatrix} \begin{bmatrix} r-t \\ s \end{bmatrix}$, we have

$$f(r, s, t) = \frac{\begin{bmatrix} r-s \\ r-t \end{bmatrix}}{\begin{bmatrix} t \\ s \end{bmatrix}} \left\{ \begin{bmatrix} r-t \\ s \end{bmatrix} - q^{s(r-2t)} \begin{bmatrix} t \\ s \end{bmatrix} \right\}$$

$$= \frac{\begin{bmatrix} r-s \\ r-t \end{bmatrix}}{\begin{bmatrix} t \\ s \end{bmatrix}} W_s(r-t, r-2t) \geq 0. \quad [\ ]$$

Using formulas (1) and (8) and Lemma 7, we can present a necessary condition on the Whitney numbers such that there exists a nested chain decomposition in $C[n, k]$ similar to that in $C(n, k)$ obtained by the bracketing construction.

Theorem 4 $W_0 \leq W_n \leq W_1 \leq W_{n-1} \leq \cdots \leq W_{n-N+1} \leq W_N$.

Proof It suffices to show that for $m < N$, $W_m \leq W_{n-m} \leq W_{m+1}$.

For $m < N$, by formula (1) and Lemma 7 it follows that

$$W_{n-m} - W_m = \left\{ \begin{bmatrix} n \\ n-m \end{bmatrix} - q^{k(n-m)} \begin{bmatrix} n-k \\ n-m \end{bmatrix} \right\} - \left\{ \begin{bmatrix} n \\ m \end{bmatrix} - q^{km} \begin{bmatrix} n-k \\ m \end{bmatrix} \right\}$$

$$= q^{km} \left\{ \begin{bmatrix} n-k \\ m \end{bmatrix} - q^{k(n-2m)} \begin{bmatrix} n-k \\ n-m \end{bmatrix} \right\} \geq 0$$

On the other hand, from formula (8) and Lemma 7 it follows that

$$W_{m+1} - W_{n-m} = \sum_{i=1}^{k} q^{(i-1)(m+1)} \begin{bmatrix} n-i \\ m \end{bmatrix} - \sum_{i=1}^{k} q^{(i-1)(n-m)} \begin{bmatrix} n-i \\ n-m-1 \end{bmatrix}$$

$$= \sum_{i=1}^{k} q^{(i-1)(m+1)} \left\{ \begin{bmatrix} n-i \\ m \end{bmatrix} - q^{(i-1)(n-1-2m)} \begin{bmatrix} n-i \\ n-1-m \end{bmatrix} \right\} \geq 0.$$

The proof is then completed. [ ]

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一类子空间格

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摘 要

令 $V_n(q)$ 是具 q 个元素的有限域上的 n 维向量空间. $C[n,k]$ 是 $V_n(q)$ 中与某 k 维子空间相交不为零空间之子空间全体按包含关系所成偏序集, W_m 为其Whitney 数 $(0 \le m \le n)$. 本文证明了 $C[n,k]$ 具 Sperner 性质和单峰性质. 进一步地, $W_m^2 - qW_{m-1}W_{m+1}$ 作为 q 的多项式具有非负系数, 并且 $W_0 \le W_n \le W_1 \le W_{n-1} \le W_2 \le \cdots$.