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The Twisted Atiyah-Singer Operators (II)*

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Abstract: In this paper we show that from the Dolbeault operator we can get a twisted Atiyah-Singer operator with the same leading symbols. In particular, the Dolbeault operator on a symplectic manifold is a twisted Atiyan-Singer operator.

Keywords: almost complex manifold, Dolbeault operator, twisted Atiyah-Singer operator, index theorem.

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We have shown in [1] that the de Rham and the Signature operators on a Riemannian manifolds are essentially the twisted Atiyah-Singer operators. In this paper we study the Dolbeault operators on almost complex manifolds. We show that from the Dolbeault operator we can get a twisted Atiyah-Singer operator with the same leading symbols. We also show that the Dolbeault operator on a symplectic manifold is a twisted Atiyah-Singer operator. Then index theorem and the Lefschetz fixed point formulas of the Dolbeault operators can be derived from the corresponding theorems of twisted Atiyah-Singer operators.

1. Algebraic Preliminaries

Let V be a $2n$-dimensional real vector space with Euclidean inner product $\langle,\rangle$ and a complex structure J. Assume that the inner product is preserved by J. Let $E_1, \cdots, E_n, E_{\bar{1}}, \cdots, E_{\bar{n}}$ be an orthonormal basis of V such that $JE_i = E_{\bar{i}}$ and $\omega_1, \cdots, \omega_n, \omega_{\bar{1}}, \cdots, \omega_{\bar{n}}$ be their dual basis, $J\omega_i = -\omega_{\bar{i}}$. Define

$$Z_i = \frac{1}{\sqrt{2}}(E_i - \sqrt{-1}E_{\bar{i}}), \quad Z_{\bar{i}} = \frac{1}{\sqrt{2}}(E_i + \sqrt{-1}E_{\bar{i}}),$$

$$\Omega_{\bar{i}} = \frac{1}{\sqrt{2}}(\omega_i - \sqrt{-1}\omega_{\bar{i}}), \quad \Omega_i = \frac{1}{\sqrt{2}}(\omega_i + \sqrt{-1}\omega_{\bar{i}}).$$

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The subspaces of $\wedge(V^*)\otimes\mathbf{C}$ generated by $\{\Omega_i\}$ and $\{\Omega_{\bar i}\}$ are denoted by $\wedge^{*,0}(V)$ and $\wedge^{0,*}(V)$ respectively. The frames $\{E_a\}$ and $\{\omega_a\}$ are called the S-basis, $\{Z_a\}$ and $\{\Omega_a\}$ are called the U-basis of V. In this paper we use the following notations:

$$a, b, \cdots \in \{1, \cdots, n, \bar 1, \cdots, \bar n\},\quad i, j, \cdots \in \{1, \cdots, n\},$$

$\bar a = \bar i$ if $a = i$; $\bar a = i$ if $a = \bar i$.

Define $L_a = \frac{1}{\sqrt{2}}\{(\omega_a + \sqrt{-1}J\omega_a) - i(E_a + \sqrt{-1}JE_a)\}$. $\{L_a\}$ act on the left of $\wedge^{0,*}(V)$. Note that $L_i = \Omega_{\bar i} - i(Z_{\bar i}), L_{\bar i} = \sqrt{-1}(\Omega_{\bar i} + i(Z_{\bar i}))$.

Lemma 1.1 *$L_aL_b + L_bL_a = -2\delta_{ab}$. Then $\{L_a\}$ generate a Clifford algebra $\mathbf{C}l(2n)$.*

Let $g: V^*\to V^*$ be an isometry such that $g\circ J = J\circ g$. If $g(\Omega_{\bar i}) = \sum G_{ij}\Omega_{\bar j}$, then $(G_{ij})\in U(n)$. Since the exponential map $\exp: u(n)\to U(n)$ is an epimorphism, we can set $G = \exp\Theta$, $\Theta = \Theta_1 + \sqrt{-1}\Theta_2$, where Θ_1 and Θ_2 are real matrices. The matrix $\Theta^* = \begin{pmatrix}\Theta_1 & \Theta_2\\ -\Theta_2 & \Theta_1\end{pmatrix}$ is the realization of Θ, $\exp\Theta^*\in SO(2n)$. The map g induces a homomorphism g^* on $\wedge^{0,*}(V)$, $g^*(\Omega_{\bar i_1}\wedge\cdots\wedge\Omega_{\bar i_k}) = g(\Omega_{\bar i_1})\wedge\cdots\wedge g(\Omega_{\bar i_k})$.

Lemma 1.2 $g^* = \exp(\frac{1}{2}\mathrm{tr}\Theta)\exp(\frac{1}{4}\sum\Theta^*_{ab}L_aL_b)$.

Proof It is easy to see that the lemma is true when Θ is a diagonal matrix and the expression of g^* is independent of the choice of U-frames. □

The Lemma 1.2 gives a representation $\rho: U(n)\to \mathrm{Spin}_c(2n)$.

Let $\mathbf{C}l_{2n}$ be the Clifford algebra. Let $e_1, \cdots, e_n, e_{\bar 1}, \cdots, e_{\bar n}$ be an orthonormal basis of Euclidean space $\mathbf{R}^{2n}$. Define $g_i = \frac{1}{2}(e_i - \sqrt{-1}e_{\bar i})$, $\bar g_i = \frac{1}{2}(e_i + \sqrt{-1}e_{\bar i})$. Let Δ be an irreducible module over $\mathbf{C}l_{2n}$ generated by $\bar g_1\cdots\bar g_n$, $\Delta = \mathbf{C}l_{2n}\cdots\bar g_1\cdots\bar g_n$. Define a homomorphism ρ: $\wedge^{0,*}(V)\to\Delta\otimes\mathbf{C}$ by: $\rho(\Omega_{\bar i_1}\wedge\cdots\wedge\Omega_{\bar i_k}) = g_{i_1}\cdots g_{i_k}\bar g_1\cdots\bar g_n\otimes 1$.

Lemma 1.3 *For any $\xi\in\wedge^{0,*}(V)$, we have $\rho(g^*\xi) = \rho(g^*)\rho(\xi)$, where homomorphism $\rho(g^*)$: $\Delta\otimes\mathbf{C}\to\Delta\otimes\mathbf{C}$ is defined by $\rho(L_a) = e_a$.*

Proof We need only to verify the following two cases (compare with [2], p. 262):

$$\rho(L_1\cdots\Omega_{\bar i_1}\wedge\cdots\wedge\Omega_{\bar i_k}) = e_1\rho(\Omega_{\bar i_1}\wedge\cdots\wedge\Omega_{\bar i_k}) = \begin{cases}-g_{i_2}\cdots g_{i_k}*\bar g_1\cdots\bar g_n, & i_1 = 1;\\ g_1g_{i_1}\cdots g_{i_k}\bar g_1\cdots\bar g_n, & i_1>1,\end{cases}$$

$$\rho(L_{\bar 1}\cdots\Omega_{\bar i_1}\wedge\cdots\wedge\Omega_{\bar i_k}) = e_{\bar 1}\rho(\Omega_{\bar i_1}\wedge\cdots\wedge\Omega_{\bar i_k}) = \begin{cases}\sqrt{-1}g_{i_2}\cdots g_{i_k}\bar g_1\cdots\bar g_n, & i_1 = 1;\\ \sqrt{-1}g_1g_{i_1}\cdots g_{i_k}\bar g_1\cdots\bar g_n, & i_1>1,\end{cases}$$

where $i_1<\cdots<i_k$. □

2 The $\bar\partial$-operators

Let M be a compact almost complex manifold of dimension $2n$ and $\langle,\rangle$ be a Hermitian metric on M, J the complex structure. Let ∇ be the Levi-Civita connection on M, ∇ be an almost complex connection defined by

$$\nabla_XY = \frac{1}{2}(\nabla_XY - J(\nabla_X(JY))),\quad X, Y\in\Gamma(TM\otimes\mathbf{C}).$$

Denote $A^e(M) = \Gamma(\wedge^{0,\mathrm{even}}(M))$, $A^o(M) = \Gamma(\wedge^{0,\mathrm{odd}}(M))$.

Let $\bar\partial$ be the restriction of the exterior differential operator d on the subspace $A^{0,*}(M)$ and $\bar\delta$ be its adjoint. Then we have a Dolbeault operator: $D = \bar\partial + \bar\delta: A^e(M) \to A^o(M)$.

Similar to §1, let $E_1, \cdots, E_n, E_{\bar 1}, \cdots, E_{\bar n}$ be local S-frame fields on M and $\omega_1, \cdots, \omega_n, \omega_{\bar 1}, \cdots, \omega_{\bar n}$ be their dual, $\{\Omega_a\}$ and $\{Z_a\}$ be the related U-frames.

Lemma 2.1 *As operators on* $A^{0,*}(M)$,

$$\bar\partial = \sum_j \Omega_{\bar j} \nabla_{Z_{\bar j}}, \quad \bar\delta = -\sum_j (Z_{(\bar j)} \nabla_{Z_j} + \sum_{k,j} \langle \nabla_{Z_{\bar k}} Z_k, Z_j \rangle i(Z_{\bar j}).$$

Proof See for example Yu[3]. □

As in §1, define operators L_a on $\wedge^{0,*}(M)$ by

$$L_j = \Omega_{\bar j} - i(Z_{\bar j}), \quad L_{\bar j} = \sqrt{-1}(\Omega_{\bar j} + i(Z_{\bar j})).$$

Proposition 2.2 *The operator D can be expressed by*

$$D = \frac{1}{\sqrt{2}} \sum_a L_a \{\check{E}_a + \frac{1}{4}\sum_{b,c} \langle \nabla_{E_a} E_b, E_c \rangle L_b L_c - \frac{\sqrt{-1}}{2} \sum_j \langle \nabla_{E_a} E_{\bar j}, E_j \rangle +$$
$$\frac{1}{8}\sum_{b,c} \langle (\nabla_{E_a} J) E_b, J E_c \rangle L_b L_c + \frac{1}{4}\sum_b \langle (\nabla_{E_b} J) E_b, J E_a + \sqrt{-1} E_a \rangle \}.$$

Proof From Lemma 2.1, we have

$$\begin{aligned} D &= \bar\partial + \bar\delta = \sum \Omega_{\bar i} \wedge \{\check{Z}_{\bar i} - \sum \langle \nabla_{Z_i} Z_{\bar j}, Z_k \rangle \Omega_{\bar j} i(Z_{\bar k})\} - \\ &\quad \sum i(Z_{\bar i})\{\check{Z}_i - \sum \langle \nabla_{Z_i} Z_{\bar j}, Z_k \rangle \Omega_{\bar j} i(Z_{\bar k})\} + \sum \langle \nabla_{Z_{\bar k}} Z_k, Z_j \rangle i(Z_{\bar j}) \\ &= \frac{1}{\sqrt{2}} \sum L_a \{\check{E}_a - \sum \langle \nabla_{E_a} Z_{\bar j}, Z_k \rangle \Omega_{\bar j} i(Z_{\bar k})\} + \sum \langle \nabla_{Z_{\bar k}} Z_k, Z_j \rangle i(Z_{\bar j}). \end{aligned}$$

Similar to Yu [3], we have

$$\sum \langle \nabla_{E_a} Z_{\bar j}, Z_k \rangle \Omega_{\bar j} i(Z_{\bar k}) = -\frac{1}{4}\sum \langle \nabla_{E_a} E_b, E_c \rangle L_b L_c + \frac{\sqrt{-1}}{2} \langle \nabla_{E_a} E_{\bar j}, E_j \rangle,$$

and

$$\begin{aligned} \sum \langle \nabla_{Z_{\bar k}} Z_k, Z_j \rangle i(Z_{\bar j}) &= -\frac{1}{4\sqrt{2}} \sum \langle (\nabla_{E_b}(E_b - \sqrt{-1} J E_b), E_a - \sqrt{-1} J E_a \rangle L_a \\ &= \frac{1}{4\sqrt{2}} \sum \langle (\nabla_{E_b} J) E_b, J E_a + \sqrt{-1} E_a \rangle L_a. \end{aligned}$$

The proposition follows from

$$\langle \nabla_{E_a} E_b, E_c \rangle = \langle \nabla_{E_a} E_b, E_c \rangle + \frac{1}{2} \langle (\nabla_{E_a} J) E_b, J E_c \rangle$$

and

$$\sum_j \langle \nabla_{E_a} E_{\bar{j}}, E_j \rangle = \sum_j \langle \nabla_{E_a} E_{\bar{j}}, E_j \rangle$$ □

Corollary 2.3 *If M is a symplectic manifold, that is, the kaehler form of M is closed. Then*

$$D = \frac{1}{\sqrt{2}} \sum_a L_a \{ \check{E}_a + \frac{1}{4} \sum_{b,c} \langle \nabla_{E_a} E_b, E_c \rangle L_b L_c - \frac{\sqrt{-1}}{2} \sum_j \langle \nabla_{E_a} E_{\bar{j}}, E_j \rangle \}.$$

Proof By assumption the kaehler form $\Phi = -\sqrt{-1} \sum \Omega_i \wedge \Omega_{\bar{i}}$ is closed. As is well known,

$$J \nabla_{JX} J = \nabla_X J,$$

and

$$\langle (\nabla_X J) Y, Z \rangle + \langle (\nabla_Y J) Z, X \rangle + \langle (\nabla_Z J) X, Y \rangle = 0$$

hold for any $X, Y, Z \in \Gamma(TM)$.

From $d\Phi = 0$, we have $\langle \nabla_{Z_i} Z_j, Z_k \rangle = 0$ (see [3]). By Proposition 2.2, we need only to show $\sum \langle (\nabla_{E_a} J) E_b, J E_c \rangle L_a L_b L_c = 0$. This can be proved as follows. As remarked above, we have

$$\begin{aligned} 0 &= \sum \{ \langle (\nabla_{E_a} J) E_b, J E_c \rangle + \langle (\nabla_{E_b} J) E_c, J E_a \rangle + \langle (\nabla_{E_c} J) E_a, J E_b \rangle \} L_a L_b L_c \\ &= 3 \sum \langle (\nabla_{E_a} J) E_b, J E_c \rangle L_a L_b L_c + \sum \langle (\nabla_{E_b} J) E_c, J E_a \rangle (-2\delta_{ab} L_c + 2\delta_{ac} L_b) + \\ &\quad \sum \langle (\nabla_{E_c} J) E_a, J E_b \rangle (-2\delta_{bc} L_a + 2\delta_{ac} L_b) \\ &= 3 \sum \langle (\nabla_{E_a} J) E_b, J E_c \rangle L_a L_b L_c + 6 \sum \langle (\nabla_{E_a} J) E_a, J E_b \rangle L_b, \end{aligned}$$

and

$$\begin{aligned} \sum_a \langle (\nabla_{E_a} J) E_a, J E_b \rangle &= \sum_i (\langle (\nabla_{E_i} J) E_i, J E_b \rangle + \langle (\nabla_{JE_i} J) J E_i, J E_b \rangle) \\ &= \sum_i (\langle (\nabla_{E_i} J) E_i, J E_b \rangle - \langle J (\nabla_{JE_i} J) E_i, J E_b \rangle) = 0. \end{aligned}$$ □

Denote

$$Q = \frac{1}{8\sqrt{2}} \sum_{a,b,c} \langle (\nabla_{E_a} J) E_b, J E_c \rangle L_a L_b L_c + \frac{1}{4\sqrt{2}} \sum_{a,b} \langle (\nabla_{E_b} J) E_b, J E_a + \sqrt{-1} E_a \rangle L_a.$$

Lemma 2.4 *Q is a self adjoint operator on $A^{0,*}(M)$.*

The proof is a direct computation, so we omit it. Then $D = \sqrt{2}(D - Q) : A^e(M) \to A^o(M)$ defines a selfadjoint operator.

Let $P_{U(n)}(M)$ be the principal bundle formed by all U-frames $\{\Omega_{\bar{i}}\}$. Then we have a twisted spinor bundle

$$\Delta(M) \otimes L = P_{U(n)}(M) \times_\rho (\Delta \otimes \mathbf{C}),$$

where the representation $\rho: U(n) \to \mathrm{Spin}_c(2n)$ is defined in §1. In general, $\Delta(M)$ and the line bundle L are not defined globally, $\exp\left(\frac{1}{2}\mathrm{tr}\Theta\right)$ is determined up to ± 1. But $L \otimes L \cong \wedge^{0,n}(M)$ is well defined. Then there is a connection on $L \otimes L$ defined naturally. From Lemma 1.2 and 1.3, we have

Proposition 2.5 *The bundles* $\wedge^{0,*}(M)$ *and* $\Delta(M) \otimes L$ *are isomorphic.*

The isomorphism $\rho: \wedge^{0,*}(M) \to \Delta(M) \otimes L$ is defined by

$$\rho(\Omega_{\bar{i}_1} \wedge \cdots \wedge \Omega_{\bar{i}_k}) = g_{i_1} \cdots g_{i_k} \overline{g_1} \cdots \overline{g_n} \otimes 1.$$

Theorem 2.6 *For any* $\xi \in A^e(M)$, *we have*

$$\rho(\tilde{D}\xi) = D_L \rho(\xi),$$

where $D_L: \Gamma(\Delta^+(M) \otimes L) \to \Gamma(\Delta^-(M) \otimes L)$ *is a twisted Atiyah-Singer operator. In particular, the Dolbeault operator on a symplectic manifold is a twisted Atiyah-Singer operator.*

Proof By Lemma 1.3, we have

$$\rho(\tilde{D}) = \sum e_a\{\check{E}_a + \frac{1}{4}\sum\langle\nabla_{E_a}E_b, E_c\rangle e_b e_c - \frac{\sqrt{-1}}{2}\sum\langle\nabla_{E_a}E_{\bar{j}}, E_j\rangle\}.$$

Since $L \otimes L \cong \wedge^{0,n}(M)$, hence $\frac{1}{2}\sum\langle\nabla_{E_a}\Omega_{\bar{i}}, \Omega_i\rangle = -\frac{\sqrt{-1}}{2}\sum\langle\nabla_{E_a}E_{\bar{i}}, E_i\rangle$ defines a connection on L. Then $D_L = \rho(\tilde{D})$ is a twisted Atiyah-Singer operator. □

If E is a holomorphic vector bundle on a compact complex manifold M, we have a twisted Dolbeault operator: $D_E: \Gamma(\wedge^{o,\mathrm{even}}(M) \otimes E) \to \Gamma(\wedge^{o,\mathrm{odd0}}(M) \otimes E)$. Obviously the Proposition 2.5 and Theorem 2.6 can be generalized to such operators.

Since the leading symbols D and $\sqrt{2}D$ are the same, the index theorem of the Dolbeault operator on almost complex manifold is an easy consequence of that of the twisted Atiyah-Singer operators. If M is kaehler or symplectic, $\tilde{D} = \sqrt{2}D$ is a twisted Atiyah-Singer operator (see also [4]). Then the local index theorem of these operators can be obtained from that of the twisted Atiyah-Singer operators.

3. The Lefschetz fixed point formulas

Let f be an isometry on M which preserves the complex structure J. Then f^* maps U-frames to itself and commutes with the Dolbeault operator D. Let $\{\Omega_a\}$ and $\{Z_a\}$ be local U-frame fields on neighborhoods of x and $f(x)$ respectively, we have

$$f_*(Z_1, \cdots, Z_n)^t = B(Z_1, \cdots, Z_n)^t, \quad B \in U(n).$$

Lemma 3.1 *Restricting* f^* *on* $\wedge^{0,*}(M)$, *we have*

$$f^* = \exp\left(-\frac{1}{2}tr\Theta\right)\exp\left(-\frac{1}{4}\Theta^*_{ab}L_aL_b\right),$$

where Θ *and* L_a *are defined as in* §1.

Lemma 3.2 The cotangent map f^* commutes with the operator Q. Then f^* commutes with the operator $\widetilde{D}$.

Let $N = \cup_i N_i$ be the set of fixed points of f, each N_i be a connected totally geodesic submanifold on M. The tangent bundle TN_i and the normal bundle $\upsilon(N_i)$ also have complex structures induced from that of M. We also decompose

$$TN_i \otimes \mathbf{C} = T^{1,0}N_i \oplus T^{0,1}N_i, \upsilon(N_i) \otimes \mathbf{C} = \upsilon^{1,0}(N_i) \oplus \upsilon^{0,1}(N_i),$$

into the $\pm\sqrt{-1}$ eigenspaces of J.

Theorem 3.3 *The Lefschetz number of f is given by*

$$L_D(f) = \sum_i \int_{N_i} \frac{Td(T^{1,0}N_i)}{\det[I - \exp(\Psi + \frac{\Omega^{\perp}}{2\pi\sqrt{-1}})]},$$

where $\exp\Psi$ *and* $\Omega^{\perp}$ *are the matrices of* $f_*|_{\upsilon^{1,0}(N_i)}$ *and the curvature on* $\upsilon^{1,0}(N_i)$ *respectively.*

Proof As $f^* \circ Q = Q \circ f^*$, the Lefschetz number of f with respect to the operators D and $\widetilde{D}$ are the same (Lawson and Michelsohn [5], p. 213, Proposition 9.4). Under the isomorphism $\rho: A^{0,*}(M) \to \Gamma(\Delta(M) \otimes L)$, f^* becomes

$$\rho(f^*) = \exp(-\frac{1}{2}tr\Theta)\exp(-\frac{1}{4}\sum\Theta^*_{ab}e_ae_b).$$

The Lefschetz fixed point formula of f can be obtained from the corresponding theorem for the twisted Atiyah-Singer operators (see Zhou [1], Theorem 3.1). □

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扭化的Atiyah-Singer算子(II)

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摘 要

本文证明了从Dolbeault **算子可以得出一个扭化的**Atiyah-Singer **算子, 它与原来的算子具有相同的主象征. 特别地, 辛流形上的**Dolbeault **算子是一个扭化的**Atiyah-Singer **算子.**