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On the Reverse Order Law $(AB)^D = B^DA^D$ *

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Abstract: By using the ranks of matrices, necessary and sufficient conditions are presented for the two term reverse order law $(AB)^D = B^DA^D$ to hold.

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1. Introduction

It is well-known that the Drazin inverse has been widely applied to the theory of finite Markov chains and singular differential and difference equations [2].

In a classic paper [3], Greville gave necessary and sufficient conditions for the two term reverse order law $(AB)^+ = B^+A^+$ to hold for two complex matrices A and B. In general, the reverse order law does not hold for Drazin inverse, that is, $(AB)^D \neq B^DA^D$. Greville[3] proved that $(AB)^D = B^DA^D$ holds under the condition $AB = BA$.

In this note, we wish to derive a general necessary and sufficient condition for the two term reverse order law $(AB)^D = B^DA^D$ to hold. We shall then examine some of the special cases that have been used in the literature.

Throughout this note, all matrices are complex with order n. We shall use $R(*)$, $N(*)$, and rank$(*)$ to denote the range, null-space, and rank of $(*)$. Let index(A) be the smallest nonnegative integer l such that $N(A^l) = N(A^{l+1})$. If index$(A) = k$ and if A^D is such that

$$A^DAA^D = A^D,\ AA^D = A^DA \text{ and } A^{k+1}A^D = A^k, \tag{1.1}$$

then A^D is called the Drazin inverse of A.

Lemma 1.1 (1) $\operatorname{rank}\begin{bmatrix} A & AQ \\ PA & B \end{bmatrix} = \operatorname{rank}(A) + \operatorname{rank}(B - PAQ)$;

(2) $\operatorname{rank}(P^DAQ^D) = \operatorname{rank}(P^sAQ^t)$, $s \geq \operatorname{index}(P)$, $t \geq \operatorname{index}(Q)$;

(3) $\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}^D = \begin{bmatrix} P^D & 0 \\ 0 & Q^D \end{bmatrix}$, *and* $\operatorname{index}\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} = \max\{\operatorname{index}(P), \operatorname{index}(Q)\}$.

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2 The two term reverse order law

Theorem 2.1 *Let* index$(A)=k_1$, index$(B)=k_2$, index$(AB)=k_3$. *Then* $(AB)^D=B^DA^D$ *holds if and only if*

$$\operatorname{rank}\begin{bmatrix} 0 & 0 & 0 & A^k \\ 0 & -M^k & B^{k_2} & 0 \\ -M^kA^{k+1} & M^{2k+1} & 0 & M^k \\ B^{k_2}A^k & 0 & -B^{2k_2+1} & 0 \end{bmatrix} = \operatorname{rank}(A^{k_1}) + \operatorname{rank}(B^{k_2}) + \operatorname{rank}(M^{k_3}), \tag{2.1}$$

where $M = AB$ *and* $k = \max\{k_1, k_3\}$.

Proof Let $N \in C^{2n\times 2n}$ and let

$$N = \begin{bmatrix} B^D & B^DA^D \\ B^D & M^D \end{bmatrix}.$$

From Lemma 1.1(1), we have $\operatorname{rank}(N) = \operatorname{rank}(B^D) + \operatorname{rank}(M^D - B^DA^D)$. Then

$$\operatorname{rank}(M^D - B^DA^D) = \operatorname{rank}(N) - \operatorname{rank}(B^D) = \operatorname{rank}(N) - \operatorname{rank}(B^{k_2}). \tag{2.2}$$

Since N can be written as

$$N = \begin{bmatrix} B^D & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} B & A^D \\ I & M^D \end{bmatrix}\begin{bmatrix} B^D & 0 \\ 0 & I \end{bmatrix},$$

it follows, from Lemma 1.1(2), that

$$\operatorname{rank}(N) = \operatorname{rank}(N_1),$$

where

$$N_1 = \begin{bmatrix} B^{2k_2+1} & B^{k_2}A^D \\ B^{k_2} & M^D \end{bmatrix}.$$

Then

$$N_1 = \begin{bmatrix} B^{2k_2+1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B^{k_2} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} A^D & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} 0 & I \\ B^{k_2} & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} M^D \begin{bmatrix} 0 & I \end{bmatrix}.$$

Also, we can rewrite N_1 as the form

$$N_1 = G + PH^DQ,$$

where

$$G = \begin{bmatrix} B^{2k_2+1} & 0 \\ 0 & 0 \end{bmatrix} \in C^{2n\times 2n},\quad P = \begin{bmatrix} B^{k_2} & 0 & 0 \\ 0 & I & I \end{bmatrix} \in C^{2n\times 3n}$$

and

$$Q = \begin{bmatrix} 0 & I \\ B^{k_2} & 0 \\ 0 & I \end{bmatrix} \in C^{3n\times 2n},\quad H = \begin{bmatrix} A & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & AB \end{bmatrix} \in C^{3n\times 3n}.$$

Using Lemma 1.1(1), we get

$$\operatorname{rank}(G + PH^DQ) = \operatorname{rank}\begin{bmatrix} H^D & H^D \\ PH^D & -G \end{bmatrix} - \operatorname{rank}(H^D)$$
$$= \begin{bmatrix} H^D & H^DQ \\ PH^D & -G \end{bmatrix} - \operatorname{rank}(A^{k_1}) - n - \operatorname{rank}(M^{k_3}).$$

Applying Lemma 1.1 and noting

$$\begin{bmatrix} H^D & H^DQ \\ PH^D & -G \end{bmatrix} = \begin{bmatrix} H^D & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} H & Q \\ P & -G \end{bmatrix}\begin{bmatrix} H^D & 0 \\ 0 & I \end{bmatrix},$$

we can easily derive

$$\operatorname{rank}\begin{bmatrix} H^D & H^DQ \\ PH^D & -G \end{bmatrix} = \operatorname{rank}\begin{bmatrix} H^{2k+1} & H^kQ \\ PH^k & -G \end{bmatrix}$$
$$= n + \operatorname{rank}\begin{bmatrix} 0 & 0 & 0 & A^k \\ 0 & -M^k & B^{k_2} & 0 \\ -M^kA^{k+1} & M^{2k+1} & 0 & M^k \\ B^{k_2}A^k & 0 & -B^{2k_2+1} & 0 \end{bmatrix}$$

where $k = \max\{k_1, k_3\}$. Then

$$\operatorname{rank}(M^D - B^DA^D) = \operatorname{rank}\begin{bmatrix} 0 & 0 & 0 & A^k \\ 0 & -M^k & B^{k_2} & 0 \\ -M^kA^{k+1} & M^{2k+1} & 0 & M^k \\ B^{k_2}A^k & 0 & -B^{2k_2+1} & 0 \end{bmatrix} -$$
$$\operatorname{rank}(A^{k_1}) - \operatorname{rank}(M^{k_3}) - \operatorname{rank}(B^{k_2}), \tag{2.3}$$

this implies $M^D = B^DA^D$ holds if and only if (2.1) holds. This completes the proof of this theorem.

Remark 1 It is interesting that the condition (2.1) does not require the computation of Drazin inverses to check the validity of the two term reverse order law.

Remark 2 From Lemma 1.1(2), we can see that (2.1) is equivalent to

$$\operatorname{rank}\begin{bmatrix} 0 & 0 & 0 & A^n \\ 0 & -M^n & B^n & 0 \\ -M^nA^{n+1} & M^{2n+1} & 0 & M^n \\ B^nA^n & 0 & -B^{2n+1} & 0 \end{bmatrix}$$
$$= \operatorname{rank}(B^{k_2}) + \operatorname{rank}(A^{k_1}) + \operatorname{rank}(M^{k_3}). \tag{2.4}$$

Remark 3 One can easily extend this case to derive a sufficient and necessary condition for the triple reverse order law $(ABC)^D = C^DB^DA^D$ to hold.

Example 1 Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{index}(A) = k_1 = 2,$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{index}(B) = k_2 = 2,$$

$$M = AB = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{index}(M) = k_3 = 2.$$

From Theorem 2.1, we have $(AB)^D = B^D A^D$ even in this case $AB \neq BA$.

Corollaray 2.2[2] *If* $AB = BA$, *then* $(AB)^D = B^D A^D$.

References

[1] Ben-Israel Greville T N E. *Generalized Inverses: Theory and Applications* [M]. Wiley-Inter-Science, 1974.

[2] Campbell S L, Meyer C D. *Generalized Inverses of Linear Transfor-Mations* [M]. Pitman, London, 1979.

[3] Greville T N E. *Note on the generalized inverse of a matrix product* [J]. SIAM Rev., 1966, **8**: 518-521.

[4] Hartwig R E. *The reverse order law revisited* [J]. Linear Algebra Appl., 1986, **76**: 241-246.

关于Drazin逆逆序律的一个注记

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摘 要

本文利用矩阵间秩的关系给出Drazin逆逆序律成立的一个充分必要条件.