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Spectral Methods for Two-Dimensional Incompressible Flow*

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Abstract: We take the two-dimensional vorticity equations as models to describe spectral methods and their combinations with finite difference methods or finite element methods, which are applicable to other similar nonlinear problems. Some numerical results and error estimates of these methods are given.

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1. Introduction

The basic idea of spectral methods for partial differential equations is to approximate exact solutions by using spectral functions as bases. The methods stem from the classical Ritz-Galerkin method, which have high accuracy so called the convergence of "infinite order". But they were not widely used for a long time because of the expensive cost of computational time. However, the discovery of the Fast Fourier Transformation (FFT) and the rapid development of modern computers removed this obstacle. Although finite difference methods and finite element methods are very successful in numerical solutions of partial differential equations, it is no doubt that for some problems spectral methods are more favorable because of its high accuracy.

Spectral methods have been applied successfully to numerical simulations in science and engineering. Gottlieb and Orszag[5] summarized theoretical results and experience of applications to many practical problems. They provided numerical analysis for linear problems. Since then, spectral methods for nonlinear problems have also advanced rapidly with their applications to fluid dynamics, weather prediction, and other fields [34, 1, 35, 21, 30, 28, 29, 33, 19, 2, 9].

The spectral method, pseudospectral method, and Tau method, which are different versions of spectral methods, can be derived from the method of weighted residual. We consider here an initial-boundary value problem as follows

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$$
\begin{cases}
L u(x,t) = f(x,t), & x \in \Omega, \quad t > 0, \\
B u(x,t) = 0, & x \in \partial\Omega, \quad t \geq 0, \\
u(x,0) = u_0(x), & x \in \overline{\Omega},
\end{cases}
\tag{1.1}
$$

where Ω is a domain with the boundary $\partial\Omega$, L is a differential operator, and B is a linear boundary operator. The method of weighted residual is to find an approximation solution to (1.1) of the form

$$
u_N(x,t) = u_B(x,t) + \sum_{n=1}^{N} a_n(t)\varphi_n(x), \tag{1.2}
$$

where trial functions $\{\varphi_n(x)\}(1 \leq n \leq N)$ are linearly independent, $u_B(x,t)$ is chosen such that $u_N(x,t)$ satisfies the boundary condition, and $a_n(t)$ are determined by the following equations

$$
\int_\Omega [L u_N(x,t) - f(x,t)]\bar{w}_n(x)\,dx = 0, \qquad t > 0, n = 1, 2, \cdots, N, \tag{1.3}
$$

where the weight functions $w_n(x)$ are linearly independent, and by a similar treatment of the initial condition.

Spectral method (Galerkin approximation) Assume that $\varphi_n(x)$ satisfy the boundary condition so that $u_B(x,t) = 0$ and weight functions $w_n(x) = \varphi_n(x)$. Therefore, (1.3) leads to

$$
(L u_N(t), \varphi_n) = (f(t), \varphi_n), \qquad n = 1, 2, \cdots, N \tag{1.4}
$$

with the inner product $(u,v) = \int_\Omega u(x)\bar{v}(x)\,dx$. Sometimes, it is more convenient to describe the scheme (1.4) via a projection operator P_N. So we define a finite dimensional linear space

$$
V_N = \mathrm{Span}\{\varphi_n: n = 1, 2, \cdots, N\}, \tag{1.5}
$$

and define $P_N u \in V_N$ such that

$$
(P_N u, \varphi_n) = (u, \varphi_n), \qquad n = 1, 2, \cdots, N. \tag{1.6}
$$

It is easy to show that $P_N u$ is uniquely determined since $\{\varphi_n(x)\}$ are linearly independent. Thus we know that (1.4) is equivalent to the following scheme

$$
P_N L u_N(x,t) = P_N f(x,t). \tag{1.7}
$$

Pseudospectral method (collocation approximation) In this case, $\varphi_n(x)$ are the same as in spectral methods. But the weight functions are taken as Dirac δ functions:

$$
w_n(x) = \delta(x - x_n), \qquad n = 1, 2, \cdots, N,
$$

where $x_n \in \overline{\Omega}$ called collocation points such that $\det(\varphi_m(x_n))_{N\times N} \neq 0$. Now, (1.3) leads to

$$
L u_N(x_n, t) = f(x_n, t), \qquad n = 1, 2, \cdots, N. \tag{1.8}
$$

Also, the scheme (1.8) can be described via an interpolation operator P_C. To this end, we define $P_C u \in V_N$ such that $P_C u(x_n) = u(x_n)$, $n = 1, 2, \cdots, N$. It is easy to see that $P_C u$ is uniquely determined since $\det(\varphi_m(x_n))_{N\times N} \neq 0$. Thus, we know that (1. 8) is equivalent to the following scheme

$$P_C L u_N(x, t) = P_C f(x, t). \qquad (1.9)$$

Tau method Here we assume that $\varphi_n(x)$ are orthogonal, but need not satisfy the boundary condition. The $u_B(x, t)$ in (1. 2) is of the form

$$u_B(x, t) = \sum_{n=N+1}^{N+m} a_n(t)\varphi_n(x),$$

where m is the number of independent boundary conditions. The weight functions are taken as $w_n(x) = \varphi_n(x)$ $(n = 1, 2, \cdots, N)$. In this case, the scheme (1. 3) is read as

$$(L u_N(t), \varphi_n) = (f(t), \varphi_n), \quad n = 1, 2, \cdots, N \qquad (1.10)$$

with the m equations given by the boundary constraints. The Tau approximation scheme (1. 10) can also be described via a projection operator[5].

In this paper, we take the two-dimensional evolutionary vorticity equations as models to introduce some numerical methods related to spectral methods:

. Fourier spectral (or pseudospectral) methods

. Fourier spectral (or pseudospectral)-difference methods

. Fourier spectral (or pseudospectral)-finite element methods

. Fourier-Chebyshev spectral (or pseudospectral) methods

The first method is for problems with periodic boundary conditions and the others for ones with semi-periodic boundary conditions, which are applicable to other similar problems.

2 Fourier Spectral or Pseudospectral Methods for the Periodic Problems

For problems with periodic boundary conditions, Fourier spectral methods are powerful.

2.1 A Fourier Spectral Method

The vorticity equations are the stream function-vorticity formulation presentations of incompressible Navier-Stokes equations.

Let $\xi(x, y, t)$ and $\psi(x, y, t)$ be the vorticity and stream function respectively. $\xi_0(x, y)$ and $f_l(x, y, t)$ $(l = 1, 2)$ are given. All of them have the period 2π for the variable x and y. Let $\Omega = \{(x, y): -\pi < x, y < \pi\}$.

We consider the following problem

$$\begin{cases} \partial \xi + J(\xi, \psi) - v\nabla^2 \xi = f_1, & \text{in } \Omega \times (0, T), \\ -\nabla^2 \psi = \xi + f_2, & \text{in } \Omega \times (0, T), \\ \xi(x, y, 0) = \xi_0(x, y), & \text{in } \Omega, \end{cases} \qquad (2.1)$$

where υ is a nonnegative constant and $J(\xi, \Psi) = \partial_y \Psi \partial_x \xi - \partial_x \Psi \partial_y \xi$.

Let $(u, v) = \frac{1}{4\pi^2} \iint_{I^2} u(x, y) \bar{v}(x, y) \mathrm{d}x \mathrm{d}y$, $\|u\|^2 = (u, u)$. To fix $\Psi(x, y, t)$, we require that $(\Psi(t), 1) = 0 \ (0 \le t \le T)$. Consider the weak formulation of (2.1) as follows

$$\begin{cases} (\partial_t \xi(t), v) + (J(\xi(t), \Psi(t)), v) + \upsilon(\nabla \xi(t), \nabla v) = (f_1(t), v), & t \in (0, T), \\ (\nabla \Psi(t), \nabla v) = (\xi(t), v) + (f_2(t), v), & t \in (0, T). \end{cases} \tag{2.2}$$

Kuo Pen-yu[22] used the spectral method for the problem (2.1) and proved strict error estimates of the scheme.

For any positive integer N, set $V_N = \mathrm{Span}\{e^{i(lx+my)}: l^2 + m^2 \le N^2\}$. Let τ be the mesh step of the variable t. Denote $u^k(x, y) = u(x, y, k\tau)$ by u^k. Define

$$u_t^k = \frac{1}{\tau}(u^{k+1} - u^k), \qquad \hat{u}^k = \frac{1}{2}(u^{k+1} + u^k).$$

A Fourier spectral scheme for solving (2.1) is to find

$$\eta^k = \sum_{l^2 + m^2 \le N^2} \eta^k_{l,m} \mathrm{e}^{i(lx+my)}, \qquad \varphi^k = \sum_{l^2 + m^2 \le N^2} \varphi^k_{l,m} \mathrm{e}^{i(lx+my)}, \tag{2.3}$$

such that for any $v \in V_N$ and $k \ge 0$,

$$\begin{cases} (\eta^k_t, v) + (J(\eta^k + \delta\tau\eta^k_t, \varphi^k), v) + \upsilon(\nabla(\eta^k + \sigma\tau\eta^k_t), \nabla v) = (f_1^k, v), \\ (\nabla \varphi^k, \nabla v) = (\eta^k + f_2^k, v), \\ (\varphi^k, 1) = 0, \\ (\eta^0, v) = (\xi_0, v), \end{cases} \tag{2.4}$$

where $\delta, \sigma \ge 0$ are parameters. We point out that if $\delta = 0$ the scheme (2.4) can be solved explicitly such that

$$\eta^{k+1}_{l,m} = \frac{1}{1 + \upsilon\sigma\tau(l^2 + m^2)} \{[1 - \upsilon(1 - \sigma)\tau(l^2 + m^2)]\eta^k_{l,m} + g^k_{l,m}\} \tag{2.5}$$

where

$$g^k_{l,m} = (f_1^k - J(\eta^k, \varphi^k), \mathrm{e}^{i(l\cdot + m\cdot)}) \tag{2.6}$$

It is easy to see that the following conservation laws hold. If $f_1^k = 0$, we have

$$(\eta^k, 1) = (\eta^0, 1), \quad k \ge 0, \tag{2.7}$$

and if $\delta = \sigma = 1/2$ in addition, we have

$$\|\eta^n\|^2 + 2\upsilon\tau \sum_{k=0}^{n-1} \|\nabla \hat{\eta}^k\|^2 = \|\eta^0\|^2. \tag{2.8}$$

Kuo Pen-yu[22] point out that if we use the filtered spherical mean summation

$$\eta^k(x, y) = \sum_{l^2 + m^2 \le N^2} \left(1 - \frac{l^2 + m^2}{N^2}\right)^{\gamma} \eta^k_{l,m} \mathrm{e}^{i(lx+my)}, \quad \gamma \ge 0. \tag{2.9}$$

then we can get better results.

2.2 A Fourier Pseudospectral Method

When the spectral method is used, we have to deal with the integration such as (2.6). In order to avoid this trouble, the pseudospectral method is developed, which is easier to perform and saves the cost in computation, and is more favorable for nonlinear problems. But this method sometimes has nonlinear instability due to the aliasing. According to Kuo Pen-yu[22], the Bochner summation (2.9) could be used for eliminating these phenomena and raise the accuracy of approximate solutions. Ma He-ping and Guo Ben-yu[26] developed a Fourier pseudospectral method by using the Bochner summation for problem (2.1). Let $W_N = \mathrm{Span}\{e^{i(lx+my)}: -N \le l,m \le N\}$, $\Omega_N = \{(qh, jh): -N \le q, j \le N\}$ with $h = 2\pi/(2N+1)$. Let $P_N: L^2(\Omega) \to V_N$ be the orthogonal projection operator defined by

$$(P_N u, v) = (u, v), \qquad \forall v \in V_N \tag{2.10}$$

and $P_C: C(\bar{\Omega}) \to W_N$ be the interpolation operator by

$$P_C u(x, y) = u(x, y), \quad \forall (x, y) \in \Omega_N. \tag{2.11}$$

For $\gamma \ge 1$ and $u \in V_N$ with the coefficients $u_{l,m}$, we define a restraint operator $R = R(\gamma)$ by

$$R u(x, y) = \sum_{l^2+m^2 \le N^2} \left(1 - \left(\frac{l^2+m^2}{N^2}\right)^{\frac{\gamma}{2}}\right) u_{l,m} e^{i(lx+my)}. \tag{2.12}$$

To approximate the nonlinear term $J(u, v)$, we define $P_C = P_N P_C$ and

$$J_C(u, v) = \frac{1}{2}\{P_C(\partial_x u \partial_y v - \partial_y u \partial_x v) + \partial_x P_C(u \partial_y v) - \partial_y P_C(u \partial_x v)\}. \tag{2.13}$$

The Fourier pseudospectral scheme for solving (2.1) is to find η, φ as (2.3) such that for $k \ge 0$,

$$\begin{cases} \eta_t + R J_C(R\eta + \delta\tau R\eta_t, R\varphi) - \upsilon\nabla^2(\eta + \sigma\tau\eta_t) = P_C f_1^k, \\ -\nabla^2\varphi = \eta + P_C f_2^k, \\ (\varphi, 1) = 0, \\ \eta^0 = P_C \xi_0, \end{cases} \tag{2.14}$$

where δ, $\sigma \ge 0$ are parameters. If $\delta = 0$, then the scheme (2.14) can be solved explicitly as in (2.5) but with

$$g_{l,m}^k = (f_1^k - R J_C(R\eta, R\varphi), e^{i(l\cdot + m\cdot)})_N, \tag{2.15}$$

in which

$$(u, v)_N = \frac{1}{(2N+1)^2} \sum_{(x,y) \in \Omega_N} u(x, y)\bar{v}(x, y)$$

and which can be computed with FFT efficiently. By the fact that for $u, v, w \in V_N$,

$$(J_C(u, v), w) + (J_C(w, v), u) = 0, \tag{2.16}$$

it is easy to see that the same conservation laws as (2.7) and (2.8) hold for (2.14).

The numerical results given in Ma Heping and Guo Benyu[26] show that the restraint operator $R(\gamma)$ improves the stability of the pseudospectral method, especially in the case when the solutions of the PDE change more rapidly. The value of γ must be chosen suitably to get good results. How to choose the parameter γ suitably is relative to the smoothness of the exact solution. Generally speaking, if the exact solution changes rapidly, we should take small γ, and conversely, take large γ.

3 Fourier Spectral or Pseudospectral-Difference Methods for the Semi-Periodic Problems

Hereafter we consider the two-dimensional vorticity equations with periodic and nonperiodic boundary conditions, and assume that υ is positive.

Let $I = \{x: 0 < x < 1\}$, $\tilde{I} = \{y: 0 < y < 2\pi\}$, and $\Omega = I \times \tilde{I}$. We assume that all functions have the period 2π for the variable y and that

$$\xi(0, y, t) = \xi(1, y, t) = \Psi(0, y, t) = \Psi(1, y, t) = 0, \qquad \forall y \in \tilde{I}, \quad t \geq 0. \tag{3.1}$$

Although Fourier spectral and pseudospectral methods are favorable for periodic problems, they does not work for the problem (2.1) with (3.1). To solve it, Murdock[31,32] used Chebyshev spectral methods, and Guo Benyu and Xiong Yueshan[17] followed the idea of Guo Benyu[7] to construct a class of spectral-difference schemes. The key point is the use of a skew symmetric decomposition of the nonlinear convection terms. Then, if the parameters in the scheme are chosen suitably, the numerical solution satisfies semi discrete conservation laws and better error estimates are obtained.

3.1 A Fourier Spectral-Difference Method

Let h be the mesh spacing in the x-direction with $Mh = 1$, and let

$$I_h = \{x = jh: 1 \leq j \leq M-1\}. \tag{3.2}$$

Define

$$Du(x, y, t) = \frac{1}{h}(u(x+h, y, t) - u(x, y, t)), \quad \bar{D}u(x, y, t) = Du(x-h, y, t),$$

$$\hat{D}u(x, y, t) = \frac{1}{2}(Du(x, y, t) + \bar{D}u(x, y, t)), \quad \nabla u(x, y, t) = D\bar{D}u(x, y, t) + \partial_y^2 u(x, y, t).$$

The key problem in the construction of a reasonable scheme is to simulate as much as possible the properties of the solution of (2.1). Indeed, if $f_1 = f_2 = 0$, then

$$\iint_\Omega \xi(x, y, t)\,dx\,dy - \upsilon \int_0^t \int_{\tilde I} [\partial_x \xi(1, y, t) - \partial_x \xi(0, y, t)]\,dx\,dy = \iint_\Omega \xi_0(x, y)\,dx\,dy \tag{3.3}$$

and

$$\iint_\Omega \xi^2(x, y, t)\,dx\,dy + 2\upsilon \int_0^t \iint_\Omega [(\partial_x \xi(x, y, s))^2 + (\partial_y \xi(x, y, s))^2]\,dx\,dy\,ds = \iint_\Omega \xi_0^2(x, y)\,dx\,dy. \tag{3.4}$$

The scheme was constructed such that its solution satisfies semi-discrete conservation laws. Note that

$$\partial_y w\,\partial_x u - \partial_x w\,\partial_y u = \partial_x(\partial_y w\,u) - \partial_y(\partial_x w\,u) = \partial_y(w\,\partial_x u) - \partial_x(w\,\partial_y u).$$

Therefore, define

$$J_1(u,v) = \partial_y v \hat{D} u - \hat{D} v \partial_y u, \qquad J_2(u,v) = \hat{D}(\partial_y v u) - \partial_y(\hat{D} v u),$$

$$J_3(u,v) = \partial_y(v\hat{D}u) - \hat{D}(v\partial_y u), \quad J^{(\alpha)}(u,v) = \sum_{l=1}^{3}\alpha_l J_l(u,v),$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3), \alpha_l \geq 0$, and $\alpha_1 + \alpha_2 + \alpha_3 = 1$.

Now set

$$V_N = \mathrm{Span}\{e^{ily}: |l| \leq N\}, \tag{3.5}$$

and let $P_N: L^2(\tilde{I}) \to V_N$ be the orthogonal projection operator such that,

$$\int (P_N u - u)\bar{v}\,dy = 0, \quad \forall v \in V_N. \tag{3.6}$$

Let η and φ be the approximations to ξ and φ respectively such that

$$\eta^k(x,y) = \sum_{|l|\leq N}\eta_l^k(x)e^{ily}, \qquad \varphi^k(x,y) = \sum_{|l|\leq N}\varphi_l^k(x)e^{ily}.$$

The spectral-difference scheme for (2.1) and (3.1) is

$$\begin{cases} \eta_t^k + P_N J^{(\alpha)}(\eta^k + \delta\tau\eta_t^k, \varphi^k) - \upsilon\Delta(\eta^k + \sigma\tau\eta_t^k) = P_N f_1^k, & \text{in } I_h \times \tilde{I}, k \geq 0, \\ -\Delta\varphi^k = \eta^k + P_N f_2^k, & \text{in } I_h \times \tilde{I}, k \geq 0, \\ \eta^k(0,y) = \eta^k(1,y) = \varphi^k(0,y) = \varphi^k(1,y) = 0, & \forall y \in \tilde{I}, k \geq 0, \\ \eta^0 = P_N\xi_0, & \text{in } I_h \times \tilde{I}, \end{cases} \tag{3.7}$$

where δ and σ are parameters such that $0 \leq \delta, \sigma \leq 1$. If $\delta = \sigma = 0$, then (3.7) is an explicit scheme. Otherwise, iteration is needed to get η^k for $k \geq 1$. Assume that $f_1^k = f_2^k = 0$, we have the following conservation laws

$$(\eta^n, 1) + \tau\sum_{k=0}^{n-1}\{(\alpha_2 + \alpha_3)A(\eta^k + \delta\tau\eta_t^k, \varphi^k) + 2\upsilon S(\eta^k + \sigma\tau\eta_t^k, 1)\} = (\eta^0, 1) \tag{3.8}$$

where

$$(u,v)_h = h\sum_{x\in I_h}(u(x),v(x))_{\tilde{I}}, \qquad (u(x),v(x))_{\tilde{I}} = \frac{1}{2\pi}\int u(x,y)\bar{v}(x,y)\,dy,$$

$$\|u\|_h^2 = (u,u)_h, \qquad |u|_{1,h}^2 = \frac{1}{2}\|D u\|_h^2 + \frac{1}{2}\|\bar{D} u\|_h^2 + \|\partial_y u\|_h^2,$$

$$A(u,v) = \frac{1}{2}(u(1-h), \partial_y v(1-h))_{\tilde{I}} - \frac{1}{2}(u(h), \partial_y v(h))_{\tilde{I}},$$

$$S(u,v) = \frac{1}{2h}(u(h), v(h))_{\tilde{I}} + \frac{1}{2h}(u(1-h), v(1-h))_{\tilde{I}}.$$

Moreover, if $\alpha_1 = \alpha_2$ and $\delta = \sigma = 1/2$, then

$$\|\eta^n\|_h^2 + 2\upsilon\tau\sum_{k=0}^{n-1}\{|\hat{\eta}^k|_{1,h}^2 + S(\hat{\eta}^k, \hat{\eta}^k)\} = \|\eta^0\|_h^2. \tag{3.9}$$

Clearly, (3.8) and (3.9) are reasonable analogies of (3.3) and (3.4), respectively.

The numerical results given in Guo Benyu and Xiong Yueshan[17] show that when $\alpha_1 = \alpha_2$ and the solutions of (3.7) satisfy the semi discrete conservation laws, better results are obtained. It is also shown that good results can be got even for small N. By a skew-symmetric decomposition of the nonlinear convection term, we obtain better numerical results than by the more conventional form. But a little more work is required for calculating the Fourier coefficients of the nonlinear term.

3.2 A Fourier Pseudospectral-Difference Method

The calculation of Fourier coefficients for the nonlinear convection term takes quite a lot of time in spectral-difference scheme. To remedy this deficiency, Guo Benyu and Xiong Yueshan[18] provided a pseudospectral-difference method following the work of Guo Benyu and Zheng Jiadong[20].

We shall use the same notations as in the above section. We first introduce the points on $\tilde{I}$: $y_j = 2\pi j/(2N+1)$ $(0 \le j \le 2N)$. Let $P_C: C(\bar{\tilde{I}}) \to V_N$ be the interpolation operator such that

$$P_C u(y_j) = u(y_j), \qquad 0 \le j \le 2N. \tag{3.10}$$

For $\gamma \ge 1$, we define a restraint operator by $R = R(\gamma)$ based on the Bochner summation such that for any $u \in V_N$ with the coefficients u_l,

$$Ru(y) = \sum_{|l| \le N}\left(1 - \left|\frac{l}{N}\right|^{\gamma}\right) u_l e^{ily}. \tag{3.11}$$

To approximate the nonlinear term $J(u, v)$, we define the following nonlinear operators

$$J_{C,1}(u,v) = P_C(\partial_y v\hat{D}u - \hat{D}v\partial_y u), \quad J_{C,2}(u,v) = \hat{D}[P_C(u\partial_y v)] - \partial_y[P_C(u\hat{D}v)],$$
$$J_{C,3}(u,v) = \partial_y[P_C(v\hat{D}u)] - \hat{D}[P_C(v\partial_y u)], \quad J_C^{(\alpha)}(u,v) = \sum_{l=1}^{3}\alpha_l J_{C,l}(u,v),$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_l \ge 0$, and $\alpha_1 + \alpha_2 + \alpha_3 = 1$.

Let η and φ be the approximations to ξ and ψ respectively, where $\eta^k(x,y), \varphi^k(x,y) \in V_N$ for all $x \in I_h$ and $k \ge 0$. The pseudospectral-difference scheme for (2.1) and (3.1) is

$$\begin{cases} \eta_t^k + RJ_C^{(\alpha)}(R\eta^k + \delta\tau R\eta_t^k, R\varphi^k) - \upsilon\Delta(\eta^k + \sigma\tau\eta_t^k) = P_C f_1^k, \\ -\Delta\varphi^k = \eta^k + P_C f_2^k, \\ \eta^k(0,y) = \eta^k(1,y) = \varphi^k(0,y) = \varphi^k(1,y) = 0, \\ \eta^0 = P_C\xi_0. \end{cases} \tag{3.12}$$

If $f_1^k = f_2^k = 0$, then

$$(\eta, 1) + \tau\sum_{k=0}^{n-1}\{(\alpha_2 + \alpha_3)A(R\eta + \delta\tau R\eta_t, R\varphi) + 2\upsilon S(\eta + \sigma\tau\eta_t, 1)\} = (\eta^0, 1). \qquad (3.13)$$

If in addition $\alpha_1 = \alpha_2$ and $\delta = \sigma = 1/2$, then (3.9) holds also.

The numerical results in Guo Benyu and Xiong Yueshan[18] show the same advantages of (3.12) as those of (3.7). In particular, even we use the skew symmetric decomposition of the nonlinear convection term, the computational time is nearly the same as by the more conventional form.

4 Fourier Spectral or Pseudospectral-Finite Element Methods for the Semi-Periodic Problems

It may be hard to generalize the combined spectral-difference method to problems on non-rectangular domain. The finite element method can be successfully applied to such problems. But the convergence rate is limited by the degree of interpolation. On the other hand, the spectral and pseudospectral methods have "infinite" order accuracy if the solutions to be approximated are infinitely differentiable. But it is very difficult to use them to solve problems on non-rectangular domains. In particular, Fourier spectral or pseudospectral methods are applicable only to periodic problems. They could not be used directly to solve (2.1) and (3.1). Canuto, Maday, and Quarteroni[3] proposed a combined pseudospectral and finite element method with application to the steady problem of Navier-Stokes equations. Guo Benyu and Cao Weiming[10] constructed a spectral-finite element scheme for solving (2.1) and (3.1).

4.1 A Fourier Spectral-Finite Element Method

Let T_h be a class of regular decompositions of the interval I and satisfy the inverse assumption[4]. Let $0 = x_0 < x_1 < \cdots < x_M = 1$ are the grid points and $I_l = (x_{l-1}, x_l)$. Define $h = \max_{1\le l\le M}|x_l - x_{l-1}|$, $\bar{h} = \min_{1\le l\le M}|x_l - x_{l-1}|$, and assume that there is a constant ρ independent h such that $h \le \rho\bar{h}$. Let IP_m be the set of polynomials of order $\le m$ and

$$S^0_{m,h} = \{v: v|_{I_l} \in IP_m \text{ for } 1 \le l \le M, v \text{ is continuous and } v(0) = v(1) = 0\}.$$

Denote by Π^m the piecewise Lagrange interpolation operator of order m onto $S^0_{m,h}$, i.e., $\Pi^m u$ is the Lagrange interpolation of order m of u on each $I_l (1 \le l \le M)$ and continuous on I. V_N and P_N are defined by (3.5) and (3.6), respectively.

The spectral-finite element method for solving (2.1) and (3.1) is to find $\eta, \varphi \in S^0_{m,h} \otimes V_N$ such that for any $v \in S^0_{m,h} \otimes V_N$ and $k \ge 0$,

$$\begin{cases} (\eta_t, v) + (J(\eta + \delta\tau\eta_t, \varphi), v) + \upsilon(\nabla(\eta + \sigma\tau\eta_t), \nabla v) = (\Pi^m f_1^k, v), \\ (\nabla\varphi, \nabla v) = (\eta + \Pi^m f_2^k, v), \\ \eta^0 = \Pi^m P_N \xi_0, \end{cases} \qquad (4.1)$$

where $\delta, \sigma \ge 0$ are parameters.

The numerical results given in Guo Benyu and Cao Weiming[10] show that

(i) With the same mesh sizes, the spectral-finite element methods give better results than the fully finite element methods.

(ii) Only a relatively small N is needed to resolve the solutions in y-direction with the spectral-finite element methods. This means a great cut in computation work. This method can be easily generalized to three-dimensional semi-periodic problems, even when the domain is not rectangular.

4.2 A Fourier Pseudospectral-Finite Element Method

In comparison with spectral methods, pseudospectral methods can be implemented more efficiently. But their stability may be poor due to the aliasing. In some problems, the optimal rate of convergence in the L^2-norm can be obtained for spectral methods, but not for pseudospectral methods [29, 25].

Guo Benyu and Ma Heping[14] presented a Fourier pseudospectral-finite element scheme for problem (2.1) and (3.1). A control operator based on the Bochner summation is used to improve the stability. Let the operators P_C, $R(\gamma)$, and $J_C(u, v)$ be the same as (3.10), (3.11), and (3.12), respectively. If $u, v, w \in S^0_{m,h} \otimes V_N$, then integrating by parts, we get (2.16).

The pseudospectral-finite element scheme for solving (2.1) and (3.1) is to find $\eta, \varphi \in S^0_{m,h} \otimes V_N$ such that for any $v \in S^0_{m,h} \otimes V_N$ and $k \geq 0$,

$$\begin{cases} (\eta_t, v) + (RJ_C(R\eta + \delta\tau R\eta_t, R\varphi), v) + \nu(\nabla(\eta + \sigma\tau\eta_t), \nabla v) = (P_C f_1^k, v), \\ (\nabla\varphi, \nabla v) = (\eta + P_C f_2^k, v), \\ \eta^0 = \Pi^m P_C \xi_0, \end{cases} \tag{4.2}$$

where parameters $0 \leq \delta, \sigma \leq 1$. If $\delta = \sigma = 1/2$ and $f_1^k = 0$, we have (2.8) from (2.16).

Here the operator $J_C(\eta, \varphi)$ is constructed so that the approximation solution satisfies a conservation similar to what the solution of (2.1) satisfies. Thus, the stability is improved and the convergence order is heightened. In fact, the main of the nonlinear error vanishes, and we therefore get better error estimates. Also the control operator R is used, which improves the stability and curbs errors, and by which the L^2-optimal error estimate is obtained.

5. Fourier-Chebyshev Spectral or Pseudospectral Methods for the Semi-Periodic Problems

The accuracy of both spectral-difference methods and spectral-finite element methods is still limited due to the approximations in non-periodic directions. Guo Benyu, Ma Heping, Cao Weiming and Huang Hui[15] proposed another kind of mixed method for solving (2.1) and (3.1) by using Fourier-spectral approximation in the periodic direction and Chebyshev-spectral approximation in the non-periodic direction. The method keeps the advantage of the convergence of "infinite order".

5.1 A Fourier-Chebyshev Spectral Method

In this section, let $I=(-1,1)$, $\tilde{I}=(0,2\pi)$, and $\Omega=I\times\tilde{I}$. We assume that all functions in (2.1) have the period 2π for the variable y and

$$\xi(\pm 1,y,t)=\psi(\pm 1,y,t)=0,\quad \forall y\in\tilde{I},\quad t\geq 0. \tag{5.1}$$

Let M and N be positive integers. Define

$$V_M(I)=\{v\in IP_M: v(-1)=v(1)=0\}. \tag{5.2}$$

Let $V_N(\tilde{I})$ be the set of all real trigonometric polynomials of degree less or equal to N with the period 2π. Define

$$V_{M,N}(\Omega)=V_M(I)\otimes V_N(\tilde{I}). \tag{5.3}$$

Let $\omega(x)=(1-x^2)^{-1/2}$ and define the space

$$L^2_\omega(\Omega)=\{v \text{ is measurable}: (v,v)_\omega<\infty\} \tag{5.4}$$

equipped with the inner product and the norm

$$(u,v)_\omega=\frac{1}{4\pi}\iint_\Omega u(x,y)v(x,y)\omega(x)\,dx\,dy,\quad \|u\|^2_\omega=(u,u)_\omega. \tag{5.5}$$

Let $P_{M,N}: L^2_\omega(\Omega)\to V_{M,N}(\Omega)$ be the orthogonal projection such that

$$(u-P_{M,N}u,v)_\omega=0,\quad \forall v\in V_{M,N}(\Omega). \tag{5.6}$$

Let τ be the step of the variable t and define $u^k_{\hat{t}}=\frac{1}{2\tau}(u^{k+1}-u^{k-1})$.

The fully discrete Fourier-Chebyshev spectral scheme for solving (2.1) and (5.1) is to find $\eta,\varphi\in V_{M,N}(\Omega)$, approximating to ξ and ψ respectively, such that for any $v\in V_{M,N}(\Omega)$,

$$\begin{cases}(\eta^k_{\hat{t}},v)_\omega+(J(\eta^k,\varphi^k),v)_\omega+\frac{\upsilon}{2}a_\omega(\eta^{k+1}+\eta^{k-1},v)=(f^k_1,v)_\omega, & k\geq 1,\\ a_\omega(\varphi^k,v)=(\eta^k+f^k_2,v)_\omega, & k\geq 0,\\ \eta^1=P_{M,N}(\xi_0+\tau\partial\xi(0)), & \eta^0=P_{M,N}\xi_0,\end{cases} \tag{5.7}$$

where $a_\omega(u,v)=\frac{1}{4\pi}\iint_\Omega\nabla u(x,y)\nabla[v(x,y)\omega(x)]\,dx\,dy$, $\partial\xi(0)=\upsilon\nabla^2\xi_0-J(\xi_0,\psi(0))+f_1(0)$.

We give two examples to show the numerical results of the method introduced above.

Example 1 Let the exact solutions of (2.1) and (3.1) be

$$\xi(x,y,t)=A\exp\{B\sin(Cx+y)+\omega t\},\quad \psi(x,y,t)=A\exp\{\omega t\}(Cx+\sin y).$$

For describing the errors, we let I_h be as (3.2), $\tilde{I}_N=\{y=2\pi j/N: 0\leq j\leq N-1\}$, and define

$$E_\infty(t)=\max_{(x,y)\in I_h\times\tilde{I}_N}|\xi(x,y,t)-\eta(x,y,t)|,$$

$$E_2(t)=\left(\frac{h}{N}\sum_{(x,y)\in I_h\times\tilde{I}_N}|\xi(x,y,t)-\eta(x,y,t)|^2\right)^{1/2},$$

where $\eta(x, y, t)$ is the solution of Fourier spectral-difference (FSD) scheme (3.7) or the solution of Fourier-Chebyshev spectral (FCS) scheme (5.7). The errors of both the FSD and FCS schemes are shown in Table I for $A = C = \omega = 0.1, B = 0.01$, and $\tau = \upsilon = 0.001$.

Table I Errors for the FSD and FCS Schemes

	FSD	FCS
$t = 1$	$M = 10, N = 4$	$M = 4, N = 4$
$E_2(t)$	0.2217E-3	0.5435E-5
$E_\infty(t)$	0.6949E-3	0.6497E-5

Example 2 Let the exact solutions of (2.1) and (5.1) be

$$\xi(x, y, t) = 0.4(x^2-1)(x^2-8)\sin 2y\ e^{t/2}, -\nabla^2\psi(x, y, t) = \xi(x, y, t). \qquad (5.8)$$

Define $I_h = \{x = \cos(\pi j/M): 0 \le j \le M\}$, $\tilde{I}_N = \{y = 2\pi j/N : 0 \le j \le N-1\}$, and

$$E(u(t), v(t)) = \left(\frac{\sum_{(x,y)\in I_h\times\tilde{I}_N} |u(x, y, t) - v(x, y, t)|^2}{\sum_{(x,y)\in I_h\times\tilde{I}_N} |u(x, y, t)|^2} \right)^{1/2},$$

Let $\eta(x, y, t)$ and $\varphi(x, y, t)$ are the solutions of Fourier spectral-finite (FSF) scheme (4.1) or the solutions of Fourier-Chebyshev spectral (FCS) scheme (5.7). The errors of both the FSF and FCS schemes are shown in Table II for $\upsilon = 0.001, \tau = 0.01$.

Table II Errors for the FSF and FCS Schemes

	FSF		FCS
$t = 5$	$M = 4, N = 4$	$M = 10, N = 4$	$M = 4, N = 4$
$E(\xi(t), \eta(t))$	0.4436E-2	0.7188E-2	0.3027E-4
$E(\psi(t), \varphi(t))$	0.1592E-1	0.1455E-2	0.1687E-4

It can be seen that the results of the Fourier-Chebyshev spectral method are much better than those of the Fourier spectral-difference method or the Fourier spectral-finite element method. Very high accuracy solutions can be obtained with the Fourier-Chebyshev method by using only a small number of modes. The weakness of this method is that it can not be applied directly to the three-dimensional semi-periodic problems on non-rectangular domains.

5.2 A Fourier-Chebyshev Pseudospectral Method

For saving the work as well as keeping the convergence rate of "infinite order", Guo Benyu and Li Jian[12] developed a Fourier-Chebyshev pseudospectral method. We use the same notations as in Section 5.1. Furthermore, let $\{x_j\}$ and $\{w_j\}$ be the nodes and weights of Gauss-Lobatto integration, namely

$$x_q = \cos\frac{q\pi}{M}, \quad \text{for } 0 \le q \le M,$$

$$w_0 = w_M = \frac{\pi}{2M}, \quad w_q = \frac{\pi}{M}, \quad \text{for } 1 \le q \le M-1.$$

Also, put $y_j = 2\pi j/(2N+1)$ and define

$$\Omega_{M,N} = \{(x_q, y_j): 0 \le q \le M, \quad 0 \le j \le 2N\}. \tag{5.9}$$

We denote by P_C the interpolation from $C(\overline{\Omega})$ to $V_{M,N}(\Omega)$, i.e.,

$$P_C u(x,y) = u(x,y), \quad \text{on } \Omega_{M,N}.$$

Recently Ma Heping and Guo Benyu[27] generalized the restraint operator used in the previous sections to the Chebyshev approximation. For this mixed method, let $\gamma_1, \gamma_2 \ge 1$ and $R = R(\gamma_1, \gamma_2)$ such that if

$$v = \sum_{q=0}^{M} \sum_{|l| \le N} u_{ql} T_q(x) e^{ily}, \tag{5.10}$$

$T_q(x)$ being the Chebyshev polynomial of degree q, then

$$Ru = \sum_{q=0}^{M} \sum_{|l| \le N} u_{ql} \left(1 - \left|\frac{q}{M}\right|^{\gamma_1}\right)\left(1 - \left|\frac{l}{N}\right|^{\gamma_2}\right) T_q(x) e^{ily}. \tag{5.11}$$

For approximating the nonlinear convection term, let $J_C(u,v) = \partial_y[P_C(u\partial_x v)] - \partial_x[P_C(u\partial_y v)]$. Let η and φ be the approximations to ξ and ψ as in (5.7).

The Fourier-Chebyshev pseudospectral scheme for (2.1) and (5.1) is to find $\eta, \varphi \in V_{M,N}(\Omega)$ such that

$$\begin{cases} \eta_{\hat{t}} + RJ_C(R\eta, R\varphi) - \dfrac{v}{2}\nabla^2(\eta^{+1} + \eta^{-1}) = P_C f_1^k, \\ -\nabla^2\varphi = \eta + P_C f_2^k, \\ \eta = P_{M,N}(\xi_0 + \tau\partial_t\xi(0)), \quad \eta^0 = P_{M,N}\xi_0, \end{cases} \tag{5.12}$$

The numerical results given in Guo Benyu and Li Jian[12] show the advantages of this approach. It provides the numerical solutions with high accuracy, but needs less work than the Fourier-Chebyshev spectral method.

6. Error Estimates

Recently spectral methods, pseudospectral methods, and related mixed methods are developing successfully. Much work has been done on the numerical analysis of these methods systematically (See Canuto, Hussaini, Quarteronl, and Zang[2], and Guo Benyu[9]). In 1981, Guo Benyu adopted a Fourier spectral method for solving the K.D.V.-Burgers' equation and strictly proved the convergence (See Kuo Penyu[23] and its foot note). This is one of the earliest theoretical work of spectral methods for nonlinear problems. Later, Guo Benyu et al. generalized this technique to the R.L.W. equation, vorticity equation, Navier-Stokes equation, and the flow with low Mach number. In particular, Guo Benyu and Ma Heping[13] used such a method for the three-dimensional compressible flow with strict error estimates which is a difficult job. These work extended g-stability (i.e., generalized stability, see Guo

Benyu[8] and Griffiths[6]) to spectral and pseudospectral methods and thus provided a new framework in the error analysis of nonlinear problems

The error estimates of the schemes introduced in Sections 2--5 have been strictly proved respectively, by Kuo Penyu[22], Ma Heping and Guo Benyu[26], Guo Benyu and Xiong Yueshan[17, 18] Guo Benyu and Cao Weiming[10], Guo Benyu and Ma Heping[14], Guo Benyu, Ma Heping, Cao Weiming and Huang Hui[15], and Guo Benyu and Li Jian[12].

We now give these theoretical results For simplicity, only the main conditions and results are given. For details, we refer readers to the papers mentioned above.

First we introduce some function spaces We denote by $H^{\alpha}(\Omega)$ the Sobolev space and by $H_p^{\alpha}(\Omega)$ the subspace of $H^{\alpha}(\Omega)$ of all functions with the period 2π for the variables x and y. Let $H_p^{\beta}(\tilde{I})$ be the Sobolev space of all functions with the period 2π for the variable y and $H_{\omega}^{\alpha}(I)$ be the Sobolev space with the weight ω. Denote by $H^{\beta}(\tilde{I}, H^{\alpha}(I))$ the abstract Sobolev space[24]. We define the nonisotropic Sobolev spaces

$$H_p^{\alpha,\beta}(\Omega) = L^2(\tilde{I}, H^{\alpha}(I)) \cap H_p^{\beta}(\tilde{I}, L^2(I)), \quad H_{\omega p}^{\alpha,\beta}(\Omega) = L^2(\tilde{I}, H_{\omega}^{\alpha}(I)) \cap H_p^{\beta}(\tilde{I}, L_{\omega}^2(I)).$$

Hereafter, assume that τ, h are suitably small and M, N are large enough, and that $\tau = O(h^2)$, $\tau = O(N^{-2})$. Let C denote various positive constants dependent of the solutions of (2 1) ξ, ψ, and $f_l (l = 1, 2)$, but independent of τ, h, M, and N.

We have the following results

1 Let ξ and ψ be the solutions of (2 1) with periodic boundary conditions Assume that $\xi(t) \in H_p^{\alpha}(\Omega)$ $(\alpha > 2)$, $\psi(t) \in H_p^{\alpha+1}(\Omega)$, and $\sigma > \frac{1}{2}$ or $\tau N^2 < \frac{1}{\upsilon(1-2\sigma)}$. If η is the solution of (2 4) or the solution of (2 14), then for $n\tau \le T$, $\|\eta - \xi^n\| \le C(\tau + N^{1-\alpha})$.

2 Let ξ and ψ be the solutions of (2 1) and (3 1). Assume that

$$\xi(t), \psi(t) \in H^{1/2+\epsilon}(I, H^{\beta+1}(\tilde{I})) \cap H^{3/2+\epsilon}(I, H^{\beta}(\tilde{I}))\ (\epsilon > 0, \beta > 0), \alpha_1 = \alpha_2,$$

and $\sigma > \frac{1}{2}$ or $\tau < \frac{4h^2}{\upsilon(1-2\sigma)(9+2N^2h^2)}$. If η is the solution of (3 7) or the solution of (3 12), then for $n\tau \le T$, $\|\eta - \xi^n\|_h \le C(\tau + h^2 + N^{-\beta})$.

3 Let ξ and ψ be the solutions of (2 1) and (3 1). Assume that $\xi(t) \in H_p^{\alpha,\beta}(\Omega)$ $(\alpha, \beta > 2)$, $\psi(t) \in H_p^{\alpha+1,\beta+1}(\Omega)$, and $\sigma > \frac{1}{2}$ or $\tau(h^{-2} + N^2) < \frac{2}{\upsilon C_I(1-2\sigma)}$, where C_I is some constant. If η is the solution of (4 1) or the solution of (4 2), then for $n\tau \le T$, $\|\eta - \xi^n\| \le C(\tau + h^{\bar{\alpha}} + N^{-\beta})$, where $\bar{\alpha} = \min(\alpha, m+1)$.

4 Let ξ and ψ be the solutions of (2 1) and (5 1). Assume that $\xi(t), \psi(t) \in H_{\omega p}^{\alpha,\beta}(\Omega)$ $(\alpha, \beta > 2)$, and for some positive constants C_1 and C_2, $C_1 N \le M \le C_2 N$, $\tau = O((M+N)^{-1/4})$. If η is the solution of (5 7) or the solution of (5 12), then for $n\tau \le T$, $\|\eta - \xi^n\|_{\omega} \le C(\tau^2 + M^{-\alpha} + N^{-\beta})$.

Remark In this paper, we consider the problems with the periodic boundary condition in one direction. If, in this direction, the boundary condition is not periodic, then we should use Chebyshev spectral or pseudospectral methods instead of Fourier ones, and their combinations with other methods (See Guo Benyu and He Songnian[11], Guo Benyu, Ma Heping and He Jingyu[16].

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二维不可压缩流的谱方法

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摘　要

本文以二维涡度方程为模型, 介绍了谱方法和拟谱方法以及它们与差分方法和有限元法相结合的混合解法. 这些方法可推广应用于其它一些类似的非线性问题. 本文还给出了这些方法的某些数值例子和误差估计结果.