# Hopf 代数余作用*

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**摘 要**: 对于Hopf代数$H$上的余模代数$A$, 当$H$是有限维或幺模(unimodular)时, 存在由交叉积$A\#H^{*\mathrm{rat}}$和余不变子代数$A^{\mathrm{co}H}$构成的Morita Context. 本文论证了对于任意的Hopf代数$H$, 结果仍是成立的.

**关键词**: Hopf代数, 余模代数, Morita Context.
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设$H$是域$k$上的Hopf代数, 对于在$H$余作用下的代数$A$, 我们所关心的是构造Smash积$A\#H^{*\mathrm{rat}}$与$H$余不变子代数$A^{\mathrm{co}H}$间的Morita Contex. 文[1]对有限维的$H$建立$A\#H^*$与$A^{H^*}$间的Morita Contex. 文[2]考虑了$H$为任意维的情形, 证明了当$H$是幺模时, 可以在$A\#H^*$的一个子代数$A\#H^{*\mathrm{rat}}$与$A^{\mathrm{co}H}$之间构造一个Morita Context. 本文证明了幺模的条件是可以除去的, 因此就同时推广了[1]和[2]的结论.

## 1 准备工作

在本文中, $k$是域, $H$是域$k$上的Hopf代数, 其余乘法记为$\Delta$, 余单位为$\epsilon$, 反极为$S$, 当$S$为双射时, $\bar{S}$表示$S$的逆. 设$S^*=\mathrm{Hom}_k(S,-)$, $H^{*\mathrm{rat}}$表示$H^*$的唯一的一个最大有理正规$H^*$-子模. 由[3], $H^{*\mathrm{rat}}$按"$\leftharpoonup$"和"$\rightharpoonup$"成为$H$-$H$-双子模, 其中: $\langle h^*\leftharpoonup h, g\rangle=\langle h^*, hg\rangle$, $\langle h\rightharpoonup h^*, g\rangle=\langle h^*, gh\rangle$, $\forall h^*\in H^{*\mathrm{rat}}$, $\forall h, g\in H$.

由[4]知, 若$\int\neq 0$(或$\int\neq 0$), 则$H^{*\mathrm{rat}}$在$H^*$中稠密且$S$是双射.

设$A$是有单位元1的右$H$-余模代数, 余模结构为$\varphi: A\to A\otimes H$, $\varphi(a)=\sum_{(a)}a_{(0)}\otimes a_{(1)}$, $\forall a\in A$. 由$\varphi$诱导的$A$的左$H^*$-模结构记为"$\cdot$", 即$h^*\cdot a=\sum_{(a)}\langle h^*, a_{(1)}\rangle a_{(0)}$, $\forall h^*\in H^*$, $a\in A$. 由[5]知此时有右撞积$A\#^RH^{*\mathrm{rat}}$, 即定义为

a) 作为向量空间$A\#^RH^{*\mathrm{rat}}$是$A\otimes H^{*\mathrm{rat}}$, $a\otimes h^*$记为$a\#h^*$, $a\in A$, $h^*\in H^{*\mathrm{rat}}$.

b) 乘法为$(a\#h^*)(b\#g^*)=\sum_{(b)}ab_{(0)}\otimes(h^*\leftharpoonup b_{(1)})g^*$, $\forall a, b\in A$, $h^*, g^*\in H^{*\mathrm{rat}}$. 以

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下用$A \# H^{*\mathrm{rat}}$代替$A \# {}^{R}H^{*\mathrm{rat}}$.

设$B = A^{\mathrm{co}H} = \{a \in A \mid \rho(a) = a \otimes 1\}$, $\int(\int)$表示$H^{*}$的左(右)积分空间,显然有

$$\int \subseteq H^{*\mathrm{rat}} \left(\int \subseteq H^{*\mathrm{rat}}\right).$$

为方便起见,以下均省略"$\sum$"符号. 例$\rho(a) = \sum_{(a)} a_{(0)} \otimes a_{(1)}$简写成$\rho(a) = a_0 \otimes a_1$,其它记法均与[4]一致

## 2 模结构与Morita Context

**引理1** 任取$0 \neq \lambda \in \int$都存在$l \in G(H)$使得$\lambda g = \langle g, l\rangle\lambda, \forall g \in H^{*}$.

**证明** 任取$0 \neq \lambda \in \int$, $H^{*\mathrm{rat}}$按下列模结构和余模结构成为右$H$-Hopf模:

"$\leftharpoonup$": $H^{*\mathrm{rat}} \otimes H \to H^{*\mathrm{rat}}$, $\langle f \leftharpoonup h, m\rangle = \langle S(h) \rightharpoonup f, m\rangle, \forall f \in H^{*\mathrm{rat}}, \forall h, m \in H$.

$\rho$: $H^{*\mathrm{rat}} \to H^{*\mathrm{rat}} \otimes H$, $\rho$是由有理左$H^{*}$-模所诱导的右$H$-余模结构

由[4]对$t_r \in \int$,存在$h \in H$,使得$\lambda = h \leftharpoonup t_r = t_r \leftharpoonup Sh, \forall g \in H^{*\mathrm{rat}}$显然有$\lambda g \in \int$,不妨设$\lambda g = \alpha_g\lambda$,其中$\alpha_g \in k$(由[6], $\int \neq 0$有$\dim \int = 1$). 因为

$$\lambda g = (t_r \leftharpoonup Sh)g = t_r(g \leftharpoonup h_1) \leftharpoonup Sh_2 = t_r\langle g, h_1\rangle \leftharpoonup Sh_2 = t_r \leftharpoonup S(\langle g, h_1\rangle h_2) = t_r \leftharpoonup S(h \leftharpoonup g)$$

又因为$\alpha_g\lambda = t_r \leftharpoonup S(\alpha_g h)$,故有$t_r \leftharpoonup S(h \leftharpoonup g) = t_r \leftharpoonup S(\alpha_g h)$,所以$h \leftharpoonup g = \alpha_g h$. 又由$h \leftharpoonup g = \langle g, h_1\rangle h_2$,可设$\Delta h = l \otimes h$(不妨设$\epsilon_{(h)} \neq 0$), $h = \epsilon(h) l$,即$l = \dfrac{h}{\epsilon(h)}$,故$\Delta l = l \otimes l$,故$l \in G(H)$. 则

$$\frac{1}{\epsilon(h)}\lambda g = (t_r \leftharpoonup Sl)g = t_r(g \leftharpoonup l) \leftharpoonup Sl = t_r\langle g, l\rangle \leftharpoonup Sl = \langle g, l\rangle t_r \leftharpoonup Sl = \langle g, l\rangle \frac{1}{\epsilon(h)}\lambda,$$

即$\lambda g = \langle g, l\rangle\lambda, \forall g \in H^{*}$. □

**引理2**[6] $\forall 0 \neq \lambda \in \int, \forall h \in H$,有$\langle\lambda, h\rangle = \langle\lambda, h_2\rangle h_1$.

**引理3** 设$\lambda, l$意义同引理1,则有$\overline{S}^{*}(l \rightharpoonup \lambda) = \lambda$且$S^{*}(l \rightharpoonup \lambda) = \lambda$

**证明** 先证$l \rightharpoonup \lambda \in \int, \forall h^{*} \in H^{*}$

$$\begin{aligned}(l \rightharpoonup \lambda)h^{*} &= (l \rightharpoonup \lambda)(l \rightharpoonup l^{-1} \rightharpoonup h^{*}) = l \rightharpoonup \lambda(l^{-1} \rightharpoonup h^{*}) = l \rightharpoonup \langle l^{-1} \rightharpoonup h^{*}, l\rangle\lambda \quad (\text{由引理}1)\\ &= l \rightharpoonup \langle h^{*}, 1\rangle\lambda = \langle h^{*}, 1\rangle(l \rightharpoonup \lambda),\end{aligned}$$

故$l \rightharpoonup \lambda \in \int$

再证$\overline{S}^{*}(l \rightharpoonup \lambda) = \lambda$ 因$l \rightharpoonup \lambda \in \int$,故$\overline{S}^{*}(l \rightharpoonup \lambda) \in \int$由引理2有

$$\langle \overline{S}^{*}(l \rightharpoonup \lambda), h_2\rangle h_1 = \langle \overline{S}^{*}(l \rightharpoonup \lambda), h\rangle|_H, \forall h \in H,$$

故有

$$\begin{aligned}\langle\lambda, h\rangle\lambda &= (\lambda \leftharpoonup h)\lambda = (\lambda \leftharpoonup h_1\epsilon(h_2))\lambda = (\lambda \leftharpoonup h_1)(\lambda \leftharpoonup \epsilon(h_2)|_H)\\ &= (\lambda \leftharpoonup h_1)(\lambda \leftharpoonup \overline{S}(h_3) \leftharpoonup h_2) = \lambda(\lambda \leftharpoonup \overline{S}(h_2)) \leftharpoonup h_1 = \langle\lambda \leftharpoonup \overline{S}(h_2), l\rangle\lambda \leftharpoonup h_1\\ &= \langle \overline{S}^{*}(l \rightharpoonup \lambda), h_2\rangle\lambda \leftharpoonup h_1 = \lambda \leftharpoonup \langle \overline{S}^{*}(l \rightharpoonup \lambda), h_2\rangle h_1\end{aligned}$$

$$= \lambda \leftharpoonup \langle \overrightarrow{S^*}(l \rightharpoonup \lambda), h\rangle|_H = \langle \overrightarrow{S^*}(l \rightharpoonup \lambda), h\rangle \lambda$$

故$\langle \lambda, h\rangle = \langle \overrightarrow{S^*}(l \rightharpoonup \lambda), h\rangle$, 所以$\overrightarrow{S^*}(l \rightharpoonup \lambda) = \lambda$ □

设$(A, \varphi)$是右$H$-余模代数, 对$0 \neq \lambda \in \int$,借助于引理1的$l \in G(H)$定义$A$的左和右$A \# H^{*\mathrm{rat}}$-模结构为:"$\rightarrow$"和"$\leftarrow$"即$\forall a, b \in A, h^* \in H^{*\mathrm{rat}}$定义

$$(a \# h^*) \rightarrow b = a(h^* \cdot b), \tag{1}$$

$$a \leftarrow (b \# h^*) = \overrightarrow{S^*}(l \rightharpoonup h^*) \cdot ab, \tag{2}$$

其中"$\cdot$"是由$\varphi$诱导的$A$的左$H^*$-模结构

引理4 设$\lambda \in \int$, $l \in G(H)$意义同上, 则$A$按(1)和(2)分别为左和右$A \# H^{*\mathrm{rat}}$-模

证明 由[2]知,$A$按(1)式运算是左$A \# H^{*\mathrm{rat}}$-模 下面只证:$A$按(2)式运算是右$A \# H^{*\mathrm{rat}}$-模 $\forall a \in A, b \# h^* \in A \# H^{*\mathrm{rat}}, c \# f^* \in A \# H^{*\mathrm{rat}}$,

$$(a \leftarrow (b \# h^*)) \leftarrow (c \# f^*) = \overrightarrow{S^*}(l \rightharpoonup h^*) \cdot ab \leftarrow c \# f^* = \overrightarrow{S^*}(l \rightharpoonup f^*) \overrightarrow{S^*}(l \rightharpoonup h^*) \cdot abc.$$

又因

$$\begin{aligned} a \leftarrow ((b \# h^*)(c \# f^*)) &= a \leftarrow (bc_0 \# (h^* \leftharpoonup c_1) f^*) = \overrightarrow{S^*}(l \rightharpoonup (h^* \leftharpoonup c_1) f^*) \cdot abc_0 \\ &= \langle (h^* \leftharpoonup c_2) f^*, \overline{S}(a_1 b_1 c_1) l\rangle a_0 b_0 c_0 = \langle h^* \leftharpoonup c_3, \overline{S}(a_2 b_2 c_2) l\rangle \langle f^*, \overline{S}(a_1 b_1 c_1) l\rangle a_0 b_0 c_0 \\ &= \langle l \rightharpoonup h^*, \overline{S}(a_2 b_2)\rangle \langle l \rightharpoonup f^*, \overline{S}(a_1 b_1 c_1)\rangle a_0 b_0 c_0 \\ &= \overrightarrow{S^*}(l \rightharpoonup h^*)(\overrightarrow{S^*}(l \rightharpoonup f^*) abc, \end{aligned}$$

所以$(a \leftarrow (b \# h^*)) \leftarrow (c \# f^*) = a \leftarrow (b \# h^*)(c \# f^*)$, 即$A$是右$A \# H^{*\mathrm{rat}}$-模 □

因为$B = A^{\mathrm{co}H}$, 故$A$是自然的左和右$B$-模

引理5 $A$既是$B$-$A \# H^{*\mathrm{rat}}$-双模, 也是$A \# H^{*\mathrm{rat}}$-$B$-双模(模结构同上).

证明 由[2]知:$A$是$A \# H^{*\mathrm{rat}}$-$B$-双模 下面只证$A$是$B$-$A \# A^{*\mathrm{rat}}$-双模

由上面知$A$是左$B$-模, 右$A \# H^{*\mathrm{rat}}$-模 又因为$\forall a, b \in A, c \in B, h^* \in H^{*\mathrm{rat}}$,

$$\begin{aligned} (ca) \leftarrow (b \# h^*) &= \overrightarrow{S^*}(l \rightharpoonup h^*) \cdot cab = \langle \overrightarrow{S^*}(l \rightharpoonup h^*), c_1 a_1 b_1\rangle c_0 a_0 b_0 \\ &= \langle \overrightarrow{S^*}(l \rightharpoonup h^*), a_1 b_1\rangle c a_0 b_0 = c \langle \overrightarrow{S^*}(l \rightharpoonup h^*), a_1 b_1\rangle a_0 b_0 = c(a \leftarrow b \# h^*). \end{aligned}$$

故$A$是$B$-$A \# H^{*\mathrm{rat}}$-双模结构 □

定义 设$R, T$是环, ${}_R M_T$是左$R$-右$T$-模, ${}_T N_R$是左$T$-右$R$-模并存在以下两个双线性映射

$$[\ ,\ ]: N \otimes_R M \rightarrow T,$$

$$(\ ,\ ): M \otimes_T N \rightarrow R,$$

则六元组$\{R, M, N, T, [\ ,\ ], (\ ,\ )\}$构成一个Morita Context, 如果以下条件成立:

1) $[\ ,\ ]$是$T$-双模映射和$R$-Middle Linear 即$[mr, n] = [m, rn]$, $\forall m \in M, n \in N, r \in R$.

2) $(\ ,\ )$是$R$-双模映射和$T$-Middle Linear 即$(mt, n) = (m t, n)$, $\forall m \in M, n \in N, t \in T$.

3) $\forall m, m' \in M, n, n' \in N$ 有$m' [n, m] = (m', n) \cdot m$ 和$[n, m] \cdot n' = n \cdot (m, n')$.

引理6 设$0 \neq \lambda \in \int$, 则

1) $\lambda \rightharpoonup A \subseteq B = A^{\mathrm{co}H}$;

2) $(\lambda \leftharpoonup h) g^* = (\lambda \leftharpoonup h_1) \langle \overrightarrow{S^*}(l \rightharpoonup g^*), h_2\rangle$, $\forall h \in H, g^* \in H^*$.

证明 由[2]知1)式成立

2) 因$H^*$是右$H$-模代数, 故有

$$(\lambda\leftharpoonup h)g^* = (\lambda\leftharpoonup h_1)(g^*\leftharpoonup\epsilon(h_2)1) = (\lambda\leftharpoonup h_1)(g^*\leftharpoonup\overline{S}(h_3)h_2) = (\lambda(g^*\leftharpoonup\overline{S}(h_2)))\leftharpoonup h_1$$
$$= \langle g^*\leftharpoonup\overline{S}(h_2), l\rangle l\leftharpoonup h_1 = \langle l\rightharpoonup g^*, \overline{S}(h_2)\rangle\lambda\leftharpoonup h_1 = (\lambda\rightharpoonup h_1)\langle\overrightarrow{S^*}(l\rightharpoonup g^*), h_2\rangle. \qquad \square$$

**定理7** 设$H$是$k$上Hopf代数, $(A,\rho)$是右$H$-余模代数, $\forall 0\neq\lambda\in\int$, 定义:

$$[\ ,\ ]: A\otimes_B A\to A\# H^{*\mathrm{rat}}$$
$$a\otimes b\to ab_0\#(\lambda\leftharpoonup b_1) = (a\#\lambda)(b\#\epsilon);$$
$$(\ ,\ ): A\otimes_{A\# H^{*\mathrm{rat}}}A\to B$$
$$a\otimes b\to\lambda\bullet ab.$$

则$\{A\# H^{*\mathrm{rat}}, A, A, B, [\ ,\ ], (\ ,\ )\}$形成Morita Context.

**证明** (i) 先证$[\ ,\ ]$是$A\# H^{*\mathrm{rat}}$-双模映射. 由[2]知, $[\ ,\ ]$是左$A\# H^{*\mathrm{rat}}$-模映射. 又因$\forall a,b\in A$, $c\# h^*\in A\# H^{*\mathrm{rat}}$,

$$[a, b\leftharpoonup(c\# h^*)] = [a, \overrightarrow{S^*}(l\rightharpoonup h^*)\bullet bc] = [a, \langle\overrightarrow{S^*}(l\rightharpoonup h^*), b_1c_1\rangle b_0c_0]$$
$$= \langle\overrightarrow{S^*}(l\rightharpoonup h^*), b_1c_1\rangle[a, b_0c_0] = \langle\overrightarrow{S^*}(l\rightharpoonup h^*), b_2c_2\rangle(ab_0c_0\#(\lambda\leftharpoonup b_1c_1))$$
$$= ab_0c_0\#(\lambda\leftharpoonup b_1c_1)\langle\overrightarrow{S^*}(l\rightharpoonup h^*), b_2c_2\rangle = ab_0c_0\#(\lambda\leftharpoonup b_1c_1)h^*$$
$$= ab_0c_0\#((\lambda\leftharpoonup b_1)\leftharpoonup c_1)h^* = (ab_0\#(\lambda\leftharpoonup b_1))(c\# h^*) = [a,b](c\# h^*),$$

故$[\ ,\ ]$是$A\# H^{*\mathrm{rat}}$-双模映射.

易证$[\ ,\ ]$是内侧$B$-线性的且$(\ ,\ )$是$B$-双模映射(证明略). 下面证$(\ ,\ )$是内侧$A\# H^{*\mathrm{rat}}$-线性的(即$A\# H^{*\mathrm{rat}}$-Middle Linear).

(ii) $\forall a,b\in A$, $c\# h^*\in A\# H^{*\mathrm{rat}}$,

$$(a\leftharpoonup(c\# h^*), b) = (\overrightarrow{S^*}(l\rightharpoonup h^*)\bullet ac, b) = \langle\overrightarrow{S^*}(l\rightharpoonup h^*), a_1c_1\rangle(a_0c_0, b)$$
$$= \langle h^*, \overline{S}(a_1c_1)l\rangle\lambda\bullet a_0c_0b = \langle h^*\leftharpoonup\overline{S}(a_1c_1), l\rangle\lambda\bullet a_0c_0b = \lambda(h^*\leftharpoonup\overline{S}(a_1c_1))\bullet a_0c_0(b))$$
$$= \langle\lambda, a_1c_1b_1\rangle\langle h^*\leftharpoonup\overline{S}(a_2c_2), a_2c_2b_2\rangle a_0c_0b_0 = \langle\lambda, a_1c_1b_1\rangle\langle h^*, b_2\rangle a_0c_0b_0$$
$$= \lambda\bullet(ac(h^*\bullet b)) = (a, c(h^*\bullet b)) = (a, c\# h^*\rightharpoonup b).$$

因此$(\ ,\ )$是内侧$A\# H^{*\mathrm{rat}}$-线性的.

(iii) 最后证结合性. $\forall a,b,c\in A$,

$$[a,b]\rightharpoonup c = (ab_0\#(\lambda\leftharpoonup b_1))\rightharpoonup c = ab_0c_0\langle\lambda\leftharpoonup b_1, c_1\rangle = ab_0c_0\langle\lambda, b_1c_1\rangle$$
$$= a(\lambda\bullet(bc)) = a(b,c)$$

又

$$a\leftharpoonup[b,c] = a\leftharpoonup(bc_0\#(\lambda\leftharpoonup c_1)) = \overrightarrow{S^*}(l\rightharpoonup(\lambda\leftharpoonup c_1))\bullet abc_0$$
$$= \langle\overrightarrow{S^*}(l\rightharpoonup(\lambda\leftharpoonup c_1), a_1b_1c_1\rangle a_0b_0c_0 = \langle\lambda\leftharpoonup c_2, \overline{S}(a_1b_1c_1l\rangle a_0b_0c_0$$
$$= \langle\lambda, c_2\overline{S}(c_1)\overline{S}(a_1b_1)l\rangle a_0b_0c_0 = \langle\overrightarrow{S^*}(l\rightharpoonup\lambda), a_1b_1\rangle a_0b_0c$$
$$= (\overrightarrow{S^*}(l\rightharpoonup\lambda)\bullet ab)c = (\lambda\bullet ab)c = (a,b)c. \qquad \square$$

注1 当$H$是幺模的Hopf代数时, 上边的$l$即为1, 所得到的结论就是[2]所讨论的Morita Context的内容.

注2 当$H$是有限维Hopf代数时, $H^{*\mathrm{rat}} = H^*$, 这时所得到的结论就是[1]讨论的情况.

注3 若$A\# H^{*\mathrm{rat}}$取$A\#^l H$即左撞积(这时$A$是左$H$-模代数)

$$(a\# f)(b\# g) = a(f_1\bullet b)\# f_2\bullet g, \quad \forall a,b\in A, \forall f,g\in H.$$

相应地定义左、右$A \# {}^{l}H$-模, 则可得到完全对偶的结论 因方法完全类似, 故不作详细讨论

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# Coactions of Hopf Algebras


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**Abstract**

For a given comodule algebra $A$ over a Hopf algebra $H$, it is well known that there exists a Morita context based on the smash product $A \# H^{*rat}$ of $A$ and $H$ and the coinvariat subalgebra $A^{coH}$. We show in this paper that the result still holds for arbitrary $H$.



**Keywords** Hopf algebra, comodule algebra, morita context.