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On a Two-sided Inequality Involving Stirling's Formula *

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Abstract: This note shows that the inequality $r_n(1+\frac{1}{12n}) < n! < r_n(1+\frac{1}{12n-0.5})$ holds for all $n \geq 1$.

Key words: two-sided inequality; Stirling's formula.

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1. Introduction

It is known that an elementary and quite short proof has been given of the following identity (see [1]).

$$n! = (\frac{n}{e})^n \sqrt{2\pi n}\exp(\sum_{k=n}^{\infty}\sum_{j=2}^{\infty}\frac{(j-1)}{2j(j+1)}(\frac{-1}{k})^j). \tag{1}$$

This identity implies Stirling's formula

$$r_n < n! < r_n(1+\frac{1}{12n-1}) \tag{2}$$

for all $n > 10$, where $r_n = (n/e)^n\sqrt{2\pi n}$ (See loc. cit.)

The object of this note is to show that the inequality (2) can be replaced by the more sharp form

$$r_n(1+\frac{1}{12n}) < n! < r_n(1+\frac{1}{12n-0.5}). \tag{3}$$

Here, (3) holds for all natural numbers n. Moreover, the numerical constant 0.5 contained in the RHS of (3) is best possible.

2. Proof of (3)

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Biography: Hsu was born in 1920 (date: Sept. 23) and has been the honorary director of Math. Institute, Dalian Univ. of Technology, since 1990.

As in [1] we may denote

$$n! = r_n \cdot \exp(\xi - \xi_n), \tag{4}$$

where $(\xi - \xi_n)$ may be written as follows (cf.[1], p.7)

$$\begin{aligned} \xi - \xi_n &= \sum_{k=n}^{\infty}\sum_{j=2}^{\infty} \frac{(j-1)}{2j(j+1)}(\frac{-1}{k})^j \\ &= \sum_{k=n}^{\infty}[\frac{1}{12k^2}(1 - \frac{1}{k} + \frac{1}{k^2} - \cdots) + \sum_{j=4}^{\infty}(\frac{(j-1)}{2j(j+1)} - \frac{1}{12})(\frac{-1}{k})^j] \\ &< \sum_{k=n}^{\infty}\{\frac{1}{12k(k+1)} - \frac{1}{120}(\frac{1}{k})^4 + \frac{1}{12}(\frac{1}{k})^5(1-\frac{1}{k})^{-1}\} \\ &= \frac{1}{12}\sum_{k=n}^{\infty}\{(\frac{1}{k} - \frac{1}{k+1}) - (\frac{1}{10} - \frac{1}{k-1})(\frac{1}{k})^4\} \\ &\leq \frac{1}{12n}, \end{aligned} \tag{5}$$

where $n > 10$.

In order to verifty the inequality on the RHS of (3), let us estimate the difference

$$\begin{aligned} \log(1 + \frac{1}{12n - 0.5}) - \frac{1}{12n} &> \sum_{j=1}^{4} \frac{(-1)^{j-1}}{j}(\frac{1}{12n-0.5})^j - \frac{1}{12n} \\ &= \frac{12n^2 - 2n - (0.5)^4}{12n \cdot (12n - 0.5)^4} > 0, \quad n = 1, 2, 3, \cdots. \end{aligned} \tag{6}$$

Thus it follows from (4), (5) and (6) that the RHS of (3) holds at least for all the integers $n > 10$.

In order to justify the LHS of (3), let us rewrite $(\xi - \xi_n)$ in the form

$$\xi - \xi_n = \sum_{k=n}^{\infty}\{\frac{1}{12k(k+1)} - \frac{1}{120}(\frac{1}{k})^4 + \sum_{j=5}^{\infty}(\frac{1}{12} - \frac{(j-1)}{2j(j+1)})(-1)^{j-1}(\frac{1}{k})^j\}.$$

Note that the series contained in $\{\frac{1}{12k(k+1)} - \frac{1}{120}(\frac{1}{k})^4 + \sum_{j=5}^{\infty}(\frac{1}{12} - \frac{(j-1)}{2j(j+1)})(-1)^{j-1}(\frac{1}{k})^j\}$ is an alternating series involving terms with decreasing absolute values. Thus it is clear that (with $k \geq n > 10$) $\sum_{j=5}^{\infty}(\frac{1}{12} - \frac{(j-1)}{2j(j+1)})(-1)^{j-1}(\frac{1}{k})^j > 0$.

Consequently we may estimate $(\xi - \xi_n)$ as follows

$$\begin{aligned} \xi - \xi_n &> \sum_{k=n}^{\infty}\frac{1}{12k(k+1)} - \frac{1}{120}\sum_{k=n}^{\infty}(\frac{1}{k})^4 \geq \frac{1}{12n} - \frac{1}{120}\int_{n-1}^{\infty}(\frac{1}{x})^4 \mathrm{d}x \\ &= \frac{1}{12n} - \frac{1}{120}\cdot\frac{1}{3}(\frac{1}{n-1})^3 \geq \log(1 + \frac{1}{12n}), \end{aligned} \tag{7}$$

where the last inequality may easily be checked by the logarithmic expansion in powers of $1/(12n)$. Hence the LHS of (3) follows from (4) and (7).

Finally, it remains to verifty (3) for the first 10 positive integers $n = 1, 2, 3, \cdots, 10$. This may be done by numerical computations. Indeed, we have the following numerical table

$r_n(1+\frac{1}{12n})$	$r_n(1+\frac{1}{12n})/n!$	$n!$	$r_n(1+\frac{1}{12n-0.5})/n!$	$r_n(1+\frac{1}{12n-0.5})$
0.998982	0.99898176	$1! = 1$	1.00232284	1.002323
1.998963	0.99948143	$2! = 2$	1.00033206	2.000664
5.998327	0.99972109	$3! = 6$	1.00010164	6.000610
23.995887	0.99982863	$4! = 24$	1.00004342	24.001042
119.986154	0.99988462	$5! = 120$	1.00002236	120.002683
719.940382	0.99991720	$6! = 720$	1.00001298	720.009348
5039.68626	0.99993775	$7! = 5040$	1.00000819	5040.04129
40318.0454	0.99995152	$8! = 40320$	1.00000550	40320.2216
362865.918	0.99996119	$9! = 362880$	1.00000386	362881.402
3628684.75	0.99996824	$10! = 3628800$	1.00000282	3628810.23
39915743.4	0.99997353	$11! = 39916800$	1.00000212	39916884.6

This completes the proof of (3) for all $n \geq 1$.

Remark 1 In the widely used Handbook [2] by Richard S. Burington it is stated that "For large values of n, $r_n < n! < r_n(1+\frac{1}{12n-1})$."

Clearly, our result implies that the condition "For large values of n" is unnecessary and can be entirely omitted.

Remark 2 The double series appearing on the RHS of (1) may be replaced by a single series via the logarithmic series expansion, viz.

$$n! = (\frac{n}{e})^n \sqrt{2\pi n} \exp\{\sum_{k=n}^{\infty}((k+\frac{1}{2})\log(1+\frac{1}{k})-1)\}. \tag{8}$$

As an exact identity for $n!$ involving stirling's formula this may be the most simple expression.

3. Applications of (3)

It is easily observed that the RHS of (3) yields an asymptotic relation for large n of the form

$$n! \approx r_n(1+\frac{1}{12n}+\frac{1}{288n^2}), \quad n \to \infty. \tag{9}$$

Here the exact error term involved is of order $O(n^{-3})$. From this one may also see that the constant 0.5 appearing in (3) is the optimal value in the sense that it cannot be replaced by any smaller ones if the order of the error term $O(n^{-3})$ is required.

As a second application of (3), let us consider Catalan's number

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\frac{(2n)!}{(n!)^2}.$$

Using (3) we have

$$r_{2n} \cdot (1 + \frac{1}{24n}) < (2n)! < r_{2n} \cdot (1 + \frac{1}{24n - 0.5}). \tag{10}$$

Thus, by making use of (3) and (10) we easily obtain

$$C_n < \frac{2^{2n}}{(n+1)\sqrt{\pi n}}(1 + \frac{2}{48n - 1})(1 + \frac{1}{12n})^{-2}, \tag{11}$$

$$C_n > \frac{2^{2n}}{(n+1)\sqrt{\pi n}}(1 + \frac{1}{24n})(1 + \frac{2}{24n - 1})^{-2}. \tag{12}$$

Certainly, these inequalities (11) and (12) are valid for all the positive integers n. Also, an asymptotic expansion of C_n for large n is easily derived from (11)-(12), viz.

$$C_n \approx \frac{2^{2n}}{(n+1)\sqrt{\pi n}}(1 - \frac{1}{8n} + \frac{1}{72n^2}), \quad n \to \infty.$$

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关于含有 Stirling 公式的双边不等式

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摘 要: 本文证明了有关 $n!$ 的一个便于应用的双边不等式，它对一切自然数都成立，且当 n 变大时，上界不等式能给出误差界为 $O(n^{-3})$ 的 Stirling 渐近公式，从一定意义上说，文中的上界不等式具有最优形式，因为其中的常数 0.5 已作了最佳选择. 文末还给出了关于 Catalan 数的一个双边不等式.