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The Bandwidth of a Class of Line Graphs *

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Abstract: The bandwidth of a class of line graphs and their optimal numbering are given.

Key words: graph; bandwidth.

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1. Introduction

In bivariate spline theory, in order to solve effectively the linear equations corresponding to the global conformality condition, we can give a suitable numbering of the grid-segments of the partition so that the coefficient matrix of the linear equations is a band matrix. We call it the numbering problem of the partitions. Each partition in the numbering problem has a counterpart graph. For example, the famous partition proposed by Morgan and Scott (see Figure 1) is equivalent to the numbering of edges of the graph shown in Figure 2. In practice, one may encounter a partition whose counterpart is shown in Figure 3.

In this paper, the numbering problem of the edges of such graphs is discussed. Let $G=(V,E)$ be a given graph and $\varphi: E\to\{1,2,\ldots,|E|\}$ be a numbering of its edges. For any $v\in V$, define:

$$D_\varphi(v) = \max\{|\varphi(uv)-\varphi(vw)|: uv\in E, vw\in E\}, \quad (1)$$

$$D_\varphi(G) = \max\{D_\varphi(v): v\in V\}, \quad (2)$$

$$D(G) = \min\{D_\varphi(G): \varphi \text{ is a numbering of } E\}. \quad (3)$$

$D_\varphi(v)$ is called the belt–length of the numbering φ for vertex v, and $D_\varphi(G)$ the belt-length of the numbering φ for a graph G. $D(G)$ is called the belt–length of G.

It is easy to see that the belt-length of any numbering φ for graph G is equal to the bandwidth of the numbering for the line graph $L(G)$. So the belt–length of G is equal

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to the bandwidth of $L(G)$. The bandwidth problem has been extensively studied, but there are only a few graphs whose bandwidths are known (cf. [3] for details). In this paper, we discuss a class of special graphs which comes from spline theory. We will determine their belt-length and give an optimal numbering of the edges of graphs in the class.

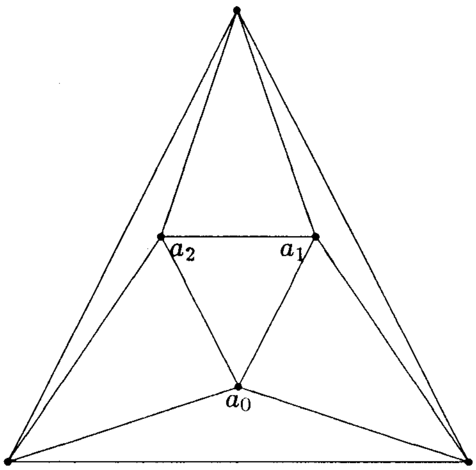

Figure 1

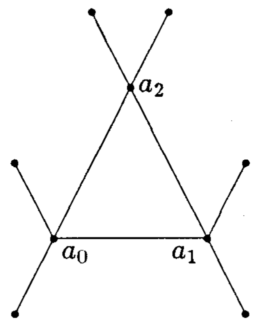

Figure 2

For any positive integers $k \geq 1$ and $s \geq 3$, the *Morgan-Scott Graph,* $G(k,s)$ is defined as follows:

(i) $G(k,s)$ is a simple graph with ks vertices;

(ii) The set of the interior vertices of $G(k,s)$ forms a cycle of length s, and the degree of every element of the set is $k+1$.

Here, the set of interior vertices of a graph consists of all its vertices v such that the degree of v is greater than one.

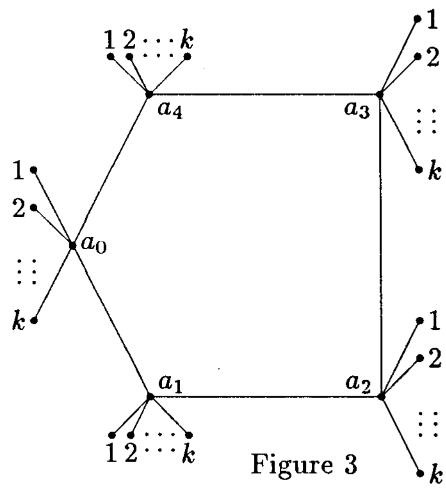

Figure 3

It is easy to see from the definition that $G(k,s)$ is a connected graphs with ks edges. Under this definition, the graphs in Figure 2 and Figure 3 are known as $G(3,3)$ and $G(k,5)$ respectively. The main result of this paper is following:

Theorem *Let $G(k,s)$ be a Morgan-Scott graph. Then*

$$D(G(k,s)) = \begin{cases} 2k, & s = 2n+1 \text{ and } k = n, \\ 2k - [\frac{k+1}{n+1}], & s = 2n+1 \text{ and } k \neq n, \\ 2k - [\frac{k}{n}], & s = 2n. \end{cases} \qquad (4)$$

In the next section, we show that the right hand side of (4) is a lower bound of $D(G(k,s))$, and, in Section 3, we present a numbering of $G(k,s)$ which achieves the lower bound, establishing the theorem.

2. A lower bound of the belt-length for $G(k,s)$

We first give a upper bound of $D(G(k,s))$ by presenting a concrete numbering of the edges:

Proposition 1 *Let $k \geq 1$ and $s \geq 3$ be any two integers. Then $D(G(k,s)) \leq 2k$.*

Proof Let $\mathrm{int}(G(k,s)) = \{a_0, \ldots, a_{s-1}\}$ which forms the cycle $(a_0, \ldots, a_{s-1})$. Let E be the set of all edges of $G(k,s)$, and $E_i = \{ua_i \in E : d(u) = 1\}$. Then E has a partition as follows:

$$E = (\bigcup_{i=0}^{s-1} E_i) \cup \{a_i a_{i+1} : i = 0, 1, \ldots, s-1\}, \tag{5}$$

where $a_s = a_0$. Let $A = \{1, \ldots, k-1\}$, and $a + A$ be the set $\{a+1, \ldots, a+k-1\}$. Then the numbering φ is given by

$$\varphi(a_i a_{i+1}) = \begin{cases} (2i+1)k, & 0 \leq i \leq [\frac{s-1}{2}], \\ 2(s-i)k, & [\frac{s-1}{2}] < i \leq s-1, \end{cases} \tag{6}$$

and

$$\varphi(E_i) = \begin{cases} A, & i = 0, \\ \varphi(a_{i-1}a_i) + A, & 0 < i \leq [\frac{s}{2}], \\ \varphi(a_i a_{i+1}) + A, & [\frac{s}{2}] < i \leq s-1. \end{cases} \tag{7}$$

It is clear that the belt-length of φ is $2k$.

Proposition 2 *Let $G(k,s)$ be a Morgan-Scott graph, Then*

$$D(G(k,s)) \geq \begin{cases} 2k, & s = 2n+1 \text{ and } k = n, \\ 2k - [\frac{k+1}{n+1}], & s = 2n+1 \text{ and } k \neq n, \\ 2k - [\frac{k}{n}], & s = 2n. \end{cases} \tag{8}$$

Proof We still adopt the notations in Proposition 1 and set $\Gamma(v) = \{uv : uv \in E\}$ for any vertex v of $G(k,s)$. let φ be any numbering of E, that is, φ is a bijection from E into $\{1, 2, \ldots, ks\}$. Without loss of generality, we may suppose that $\varphi^{-1}(1) \in \Gamma(a_0)$. For every t with $0 < t \leq [(s-1)/2]$, let $W_t = \{a_0, a_1, \ldots, a_t, a_{s-1}, \ldots, a_{s-t}\}$, and,

$$\Gamma_t = \bigcup_{u \in W_t} \Gamma(u). \tag{9}$$

Then $h = \max(\varphi(\Gamma_t)) \geq |\Gamma_t|$ holds. Since $\varphi^{-1}(h) \in \Gamma_t$, there exists a number m, $0 \leq m \leq t$

or $1 \le s - m \le t$, such that $\varphi^{-1}(h) \in \Gamma(a_m)$. If $0 \le m \le t$, then

$$\begin{aligned} D_\varphi(a_0) &\ge |\varphi(a_0a_1) - 1|, \\ D_\varphi(a_1) &\ge |\varphi(a_1a_2) - \varphi(a_0a_1)|, \\ &\cdots \quad \cdots \\ D_\varphi(a_{m-1}) &\ge |\varphi(a_{m-1}a_m) - \varphi(a_{m-1}a_{m-2})|, \\ D_\varphi(a_m) &\ge |h - \varphi(a_{m-1}a_m)|. \end{aligned} \tag{10}$$

On the other hand,

$$D_\varphi(G(k,s)) \ge \max(D_\varphi(a_0), \ldots, D_\varphi(a_m)) \ge \frac{\sum_{i=0}^{m} D_\varphi(a_i)}{m+1} \ge \frac{h-1}{m+1}.$$

Since $0 \le m \le t$ and $h \ge |\Gamma_t|$, we have

$$D_\varphi(G(k,s)) \ge \frac{|\Gamma_t| - 1}{t+1}. \tag{11}$$

Similarly, we can prove the validity of (11) for $1 \le s - m \le t$.

When $s = 2n$, we take $t = [\frac{s-1}{2}] = n - 1$, then $|\Gamma_{n-1}| = 2nk - k + 1$. So

$$D_\varphi(G(k,s)) \ge \frac{2nk - k}{n} = 2k - \frac{k}{n}. \tag{12}$$

Since the left side of (12) is an integer, we have that

$$D_\varphi(G(k,s)) \ge 2k - [\frac{k}{n}]. \tag{13}$$

Similarly, we can prove that

$$D_\varphi(G(k,s)) \ge 2k - [\frac{k+1}{n+1}] \tag{14}$$

holds for $s = 2n + 1$. By the arbitrariness of φ, $D(G(k,s))$ has the same lower bound as in (12) and (13).

Furthermore, when $s = 2n + 1$ and $k = n$, (13) can be improved to

$$D_\varphi(G(k,s)) \ge 2k.$$

Suppose to the contrary that there exists a numbering φ^* such that $D_{\varphi^*}(G(k,s)) = 2k - 1$. This implies that all inequalities in (10) and (11) become equalities when φ^* is substituted for φ. Suppose $\varphi^{*-1}(1) \in \Gamma(a_0)$ and $\varphi^{*-1}(ks) \in \Gamma(a_m)$. It is easy to see that $m = n$ or $m = n + 1$. By the symmetric property, we may assume that $m = n$. Thus

$$\begin{aligned} D_{\varphi^*}(a_0) &= 2k - 1 = \varphi^*(a_0a_1) - 1, \\ D_{\varphi^*}(a_i) &= 2k - 1 = \varphi^*(a_{i+1}a_i) - \varphi^*(a_ia_{i-1}) \ (i = 1, \ldots, n-1), \\ D_{\varphi^*}(a_n) &= 2k - 1 = sk - \varphi^*(a_{n-1}a_n). \end{aligned} \tag{15}$$

Let

$$\Delta_1 = \bigcup_{i=1}^{n-1} \Gamma(a_i), \quad \Delta_2 = \bigcup_{i=n+1}^{s-1} \Gamma(a_i).$$

Then $E = \Delta_1 \bigcup \Delta_2 \bigcup E_0 \bigcup E_n$. By (14), we have

$$2k \leq \varphi^*(\Delta_1) \leq sk - 2k + 1. \tag{16}$$

Consider the set $A = \{1, \ldots, k\}$. $A \bigcap \varphi(E_n)$ is an empty set, for otherwise $D_{\varphi^*}(a_n) \geq sk - k = 2nk > 2k - 1$. This together with (15) implies $A \subset \varphi(\Delta_2 \bigcup E_0)$. Since $|E_0| = k - 1$, we have $m = \min(\varphi^*(\Delta_2)) \leq k$. Consider set $L = \{2nk + 1, \ldots, sk\}$. By the same argument, $M = \max(\varphi^*(\Delta_2)) \geq 2nk + 1$. Suppose $m \in \varphi^{*-1}(\Gamma(a_i))$ and $M \in \varphi^{*-1}(\Gamma(a_j))$, where $n + 1 \geq i, j \geq s - 1$. Considering the belt-length for $a_i, \ldots, a_j$, we have

$$\begin{aligned} D_{\varphi^*}(G(k,s)) &\geq \max(D_{\varphi^*}(a_i), \ldots, D_{\varphi^*}(a_j)) \geq \frac{M - m}{|i - j| + 1} \geq \frac{2nk - k + 1}{n} \\ &= 2k - 1 + \frac{1}{n} > 2k - 1. \end{aligned}$$

This contradicts the assumption that $D_{\varphi^*}(G(k,s)) = 2k - 1$. So the proof is complete.

3. An optimal numbering

When $k < n$ or $k = n$ and $s = 2n + 1$, (6) and (7) have given an optimal numbering of $G(k,s)$. In this section, we give an optimal numbering for the remaining cases. Set

$$r = r(s,k) = \begin{cases} [\frac{k}{n}], & s = 2n \text{ and } k \geq n, \\ [\frac{k+1}{n+1}], & s = 2n+1 \text{ and } k > n. \end{cases} \tag{17}$$

It is easy to see that $ir \leq k$ holds for each positive integer s when $0 \leq i \leq n$. This implies

$$1 + (2k - r)i \in A_{2i-1} \ (\text{ for } \ 1 \leq i \leq n), \tag{18}$$

where $A_i = ik + A$ for $i = 0, 1, \ldots, s - 1$, and, $A = \{1, 2, \ldots, k\}$. When $s = 2n$ and $k \geq n$, it is easy to see

$$ir \leq k - 1 \ (\text{ for } \ 0 \leq i < s - n). \tag{19}$$

When $s = 2n + 1$ and $k > n$, we have

$$\frac{k}{n} > \frac{k+1}{n+1} \geq r.$$

This also implies that (19) is valid. Using (18), we get:

$$k + (2k - r)i \in A_{2i} \quad (\text{ for } \ 0 \leq i < s - n). \tag{20}$$

Now we may define the numbering φ as follows:

$$\varphi(a_{i-1}a_i) = \begin{cases} 1 + (2k - r)i, & 1 \leq i \leq n, \\ k + (2k - r)(s - i), & n < i \leq s, \end{cases} \tag{21}$$

and

$$\varphi(E_{s-i}) = \begin{cases} A_{2i-1} - \{\varphi(a_{i-1}a_i)\}, & 1 \le i \le n, \\ A_{2(s-i)} - \{\varphi(a_{i-1}a_i)\}, & n < i \le s. \end{cases} \quad (22)$$

It is easy to see that $D_\varphi(a_i) = 2k - r$ except that

$$D_\varphi(a_n) = \begin{cases} k + (n-1)r, & s = 2n, \\ k + nr - 1, & s = 2n+1. \end{cases}$$

In both cases $D_\varphi(a_n) \le 2k - r$. So $D_\varphi(G) = 2k - r$.

Now we have presented a numbering whose belt-length is equal to the lower bound. This means (4) holds.

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一类线图的带宽

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摘 要: 本文给出了一类线图的带宽，并给出了它的最佳编号.