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Relationship between Reflections Determined by Imaginary Roots and the Weyl Group for a Special GKM Algebra *

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Abstract: It's well known that a reflection r_α associated to every root α belongs to the Weyl group of a Lie algebra $g(A)$ of finite type. When $g(A)$ is a symmetrizable Kac-Moody algebra of indefinite type, one can define a reflection $r\alpha$ for every imaginary root α satisfying $(\alpha,\alpha) < 0$. From [3] we know $r_\alpha \in -W$ or r_α is an element of $-W$ mutiplied by a diagram automorphism. How about the relationship between reflections associated to imaginary roots and the Weyl group of a symmetrizable Generalized Kac-Moody algebra (GKM algebra for short)? We shall discuss it for a special GKM algebra in present paper (see 3). In sections 1 and 2 we introduce some basic concepts and give the set of imaginary roots of a class of rank 3 GKM algebras.

Key words: generalized Kac-Moody algebra; imaginary root system; the Weyl group; special imaginary root.

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1. Basic Concepts

Let $A = (a_{ij})_{n\times n}$ be a real $n \times n$ matrix satisfying the following conditions
(c1) $a_{ii} = 2$ or $a_{ii} \leq 0$;
(c2) $a_{ij} \leq 0$, if $i \neq j, a_{ij} \in Z$, if $a_{ii} = 2$;
(c3) $a_{ij} = 0$ if and only if $a_{ji} = 0$.
Then the Lie algebra $g(A)$ associated with A is called the generalized Kac-Moody algebra (see [1] or [2] for details).

Let $(\eta, \Pi, \Pi^\vee)$ be a realization, where $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ and $\Pi^\vee = \{\alpha_i^\vee, \cdots, \alpha_n^\vee\}$ are linear independent sets in η^* and η respectively. We use $Q = \sum_{i=1}^{n} Z\alpha_i$ to denote the root

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lattice and $Q_+ = \sum_{i=1}^{n} Z_+\alpha_i$ the positive root lattice of $g(A)$. Denote by Δ (resp. Δ_+) the root system (resp. positive root set) of $g(A)$. Let $\Pi^{re} = \{\alpha \in \Pi | a_{ii} = 2\}$ be the real simple root set, $\Pi^{im} = \{\alpha \in \Pi | a_{ii} \leq 0\}$ the imaginary simple root set, $W = \langle r_i | \alpha_i \in \Pi^{re} \rangle$ the Weyl group of $g(A)$. We define the real (resp. imaginary) root set of $g(A)$ to be $\Delta^{re} = W(\Pi^{re})$ (resp. $\Delta^{im} = \Delta \backslash \Delta^{re}$). It's clear that $\Delta_+^{re} = \Delta^{re} \cap \Delta_+$ is the positive real root set of $g(A)$, and $\Delta_+^{im} = \Delta^{im} \cap \Delta_+$ the positive imaginary root set of $g(A)$. We use $C^\vee = \{\lambda \in \eta^* 7 | \langle \lambda, \alpha_i^\vee \rangle \geq 0, \alpha_i \in \Pi^{re}\}$ to denote the dual fundamental Weyl chamber and $N = Z_+ \backslash \{0\}$ to denote the set of natural numbers.

Put $K_0 = \{\alpha \in Q_+ \backslash \{0\} | \alpha \in -C^\vee$ and $\mathrm{supp}\alpha$ is connected$\}$ and $K = K_0 \backslash \bigcup_{j \geq 2} j\Pi^{im}$. From formula (11.13.3) in [2], we know the following proposition

Proposition 1 $\Delta_+^{im} = \bigcup_{w \in W} w(K)$.

If A is symmetrizable, there exists an invertible diagonal matrix $D = \mathrm{diag}(\varepsilon_1, \cdots, \varepsilon_n)$ and a symmetric matrix $B = (b_{ij})$ such that $A = DB$. There exists a symmetric bilinear form (,) which is non-degenerate on η. We have an isomorphism $\nu : \eta \to \eta^*$ defined by

$$\langle \nu(h), h_1 \rangle = (h | h_1), \quad h, h_1 \in \eta$$

and the induced bilinear form (,) on η^*. It is clear that $\nu(\alpha_i^\vee) = \varepsilon_i \alpha_i, (\alpha_i, \alpha_j) = b_{ij}, i, j = 1, 2, \cdots, n$. From (2.1.6) in [2] we know the following proposition

Proposition 2 $\alpha_i^\vee = \begin{cases} \frac{2}{(\alpha_i, \alpha_i)} \nu^{-1}(\alpha_i), & if \ a_{ii} = 2, \\ -\frac{2}{(\alpha_i, \alpha_i)} \nu^{-1}(\alpha_i), & if \ a_{ii} = -2. \end{cases}$

2. The imaginary root system of a class of rank 3 generalized Kac-Moody algebras

Lemma 1 *Let* $A = \begin{bmatrix} 2 & -1 & -b \\ -1 & 2 & -c \\ -b & -c & -2 \end{bmatrix}$, *where* $b, c \in N$. *Then*

$$\begin{aligned} K = \{\alpha_3\} & \bigcup \{k_1\alpha_1 + k_3\alpha_3 |, 2k_1 \leq bk_3, k_1, k_3 \in N\} \bigcup \\ & \{k_2\alpha_2 + k_3\alpha_3 | 2k_2 \leq ck_3, k_2, k_3 \in N\} \bigcup \\ & \{k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 | k_1, k_2, k_3 \in N, 2k_1 \\ & \leq k_2 + bk_3, 2k_2 \leq k_1 + ck_3\}. \end{aligned}$$

Proof Since $\Pi^{re} = \{\alpha_1, \alpha_2\}$ and $\Pi^{im} = \{\alpha_3\}, C^\vee = \{\lambda \in \eta | \langle \lambda, \alpha_i^\vee \rangle \geq 0, i = 1, 2\}$. Let $\alpha = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 \in Q_+ \backslash \{0\}$. Then

$$\langle \alpha, \alpha_1^\vee \rangle = 2k_1 - k_2 - bk_3, \quad \langle \alpha, \alpha_2^\vee \rangle = 2k_2 - k_1 - ck_3.$$

Thus

$$\begin{aligned} K_0 &= \{\alpha \in Q_+ \backslash \{0\} | \alpha \in -C^\vee \text{ and } \mathrm{supp}\alpha \text{ is connected}\} \\ &= \{\alpha = \sum_{i=1}^{3} k_i\alpha_i \in Q_+ \backslash \{0\} | 2k_1 \leq k_2 + bk_3, 2k_2 \leq k_1 + ck_3\}. \end{aligned}$$

It is clear that $k_3 \neq 0$ for every $\alpha = \sum_{i=1}^{3} k_i\alpha_i \in K_0$.

If $k_1 = k_2 = 0$, then $\alpha = k_3 a_3 \in K_0 \rightleftharpoons k_3 \in N$. Hence $K_0 = (\bigcup_{j\geq 1} j\Pi^{\text{im}})\bigcup\{k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 | 2k_1 \leq k_2 + bk_3, 2k_2 \leq k_1 + ck_3, k_3 \in N, k_1$ and k_2 are not zero at the same time$\}$. Thanks to $K = K_0 \backslash \bigcup_{j\geq 2} j\Pi^{\text{im}}$, we get the proof of Lemma 1.

For the sake of simplicity, we let $p(k_1, k_2, k_3)$ denote the set of all k_1, k_2 and k_3 satisfying conditions: $2k_1 \leq k_2 + bk_3, 2k_2 \leq k_1 + ck_3, k_1, k_2, k_3 \in N, k_1$ and k_2 are not zero at the same time and $k_3 \neq 0$. Then we can describe K in Lemma 1 as follows

$$K = \{\alpha_3\}\bigcup\{k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 | p(k_1, k_2, k_3)\}.$$

Theorem 1 *Let* $A = \begin{bmatrix} 2 & -1 & -b \\ -1 & 2 & -c \\ -b & -c & -2 \end{bmatrix}$, *where* $b, c \in N$. *Then the positive imaginary root set* $\triangle_+^{\text{im}}$ *of* $g(A)$ *is*

$$\begin{aligned}&\{\alpha_3, b\alpha_1 + \alpha_3, c\alpha_2 + \alpha_3, b\alpha_1 + (b+c)\alpha_2 + \alpha_3, (b+c)\alpha_1 + c\alpha_2 + a_3,\\ &(b+c)\alpha_1 + (b+c)\alpha_2 + \alpha_3\}\bigcup\{k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3, (k_2 + bk_3 - k_1)\alpha_1 +\\ &k_2\alpha_2 + k_3\alpha_3, k_1\alpha_1 + (k_1 - k_3 + ck_3)\alpha_2 + k_3\alpha_3,\\ &(bk_3 + k_2 - k_1)\alpha_1 + (k_3(b+c) - k_1)\alpha_2 + k_3\alpha_3,\\ &((b+c)k_3 - k_2)\alpha_1 + (ck_3 - k_2 + k_1)\alpha_2 + k_3\alpha_3,\\ &(k_3(b+c) - k_2)\alpha_1 + (k_3(b+c) - k_1)\alpha_2 + k_3\alpha_3 | p(k_1, k_2, k_3)\}.\end{aligned}$$

Proof Since the Weyl gruop of $g(A)$ is

$$W = \langle r_i | \alpha_i \in \Pi^{re}\rangle = \langle r_1, r_2\rangle = \{1, r_1, r_2, r_1r_2, r_2r_1, r_1r_2r_1\},$$

we get

$$\begin{aligned}r_1(K) &= \{b\alpha_1 + \alpha_3\}\bigcup\{(k_2 + bk_3 - k_1)\alpha_1 + k_2\alpha_2 + k_3\alpha_3 | p(k_1, k_2, k_3)\},\\ r_2(K) &= \{c\alpha_2 + \alpha_3\}\bigcup\\ &\quad\{k_1\alpha_1 + (k_1 + ck_3 - k_2)\alpha_2 + k_3\alpha_3 | p(k_1, k_2, k_3)\},\\ r_1r_2(K) &= \{(b+c)\alpha_1 + c\alpha_2 + \alpha_3\}\bigcup\\ &\quad\{(k_3(b+c) - k_2)\alpha_1 + (ck_3 - k_2 + k_1)\alpha_2 + k_3\alpha_3 | p(k_1, k_2, k_3)\},\\ r_2r_1(K) &= \{b\alpha_1 + (b+c)\alpha_2 + \alpha_3\}\bigcup\\ &\quad\{bk_3 + k_2 - k_1)\alpha_1 + (k_3(b+c) - k_1)\alpha_2 + k_3\alpha_3 | p(k_1, k_2, k_3)\},\\ r_1r_2r_1(K) &= \{(b+c)(\alpha_1 + \alpha_2) + \alpha_3\}\bigcup\\ &\quad\{(k_3(b+c) - k_2)\alpha_1 + (k_3(b+c) - k_1)\alpha_2 + k_3\alpha_3 | p(k_1, k_2, k_3)\}.\end{aligned}$$

As

$$\triangle_+^{im} = K\bigcup r_1(K)\bigcup r_2(K)\bigcup r_1r_2(K)\bigcup r_2r_1(K)\bigcup r_1r_2r_1(K),$$

we obtain the proof of this theorem.

3. The relationship between reflections determined by imaginary roots and the Weyl group of $g(A)$

In this section the concept of a special imaginary root is introduced from Kac-Moody algebras to generalized Kac-Moody algebras. We also discuss the relationship between reflections determined by imaginary roots and the Weyl group of $g(A)$ (see theorem 2). As an application, some special imaginary roots are obtained.

Let $g(A)$ be a symmetrizable generalized Kac-Moody algebra, α be an imaginary root of $g(A)$. If $(\alpha,\alpha)\neq 0$, then we define a reflection on η^* by

$$r_\alpha(\lambda)=\lambda-2\frac{(\lambda,\alpha)}{(\alpha,\alpha)},\ \text{for } \lambda\in\eta^*.$$

If we set $\alpha^\vee=\frac{2}{(\alpha,\alpha)}\nu^{-1}(\alpha)$, then we know

$$\alpha_i^\vee=\begin{cases}\alpha_i^\vee, & \text{if } a_{ii}=2,\\ -\alpha_i^\vee, & \text{if } a_{ii}=-2\end{cases}$$

by Proposition 2 and we have $r_\alpha(\lambda)=\lambda-\langle\lambda,\alpha^\vee\rangle\alpha$, for $\lambda\in\eta^*$.

Definition 1 *Let α be an imaginary root of $g(A)$ which is a symmetrizable generalized Kac-Moody algebra. We call α a special imaginary root, if α satisfies the following conditions:*

(s1) $(\alpha,\alpha)\neq 0$

(s2) $r_\alpha(\Delta)=\Delta, r_\alpha(\Delta^{\rm re})=\Delta^{\rm re}, r_\alpha(\Delta^{\rm im})=\Delta^{\rm im}$

It is clear that if $r_\alpha\in -W$ then α is a special imaginary root.

Let $g(A)$ be a rank n generalized Kac-Moody algebra. We use (ij) to denote the diagram automorphism of $g(A)$ determined by exchanging indices i and j of Chevalley generators e_k and $f_k(k=1,\cdots,n)$ of $g(A)$. An induced action of (ij) on η^* is obtained natrually (see [1] for details). For example: Let $A=\begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&-2\end{bmatrix}$. We get that (12) is a diagram automorphism of $g(A)$. It is easy to see that

$$(12)W=\{(12),(12)r_1,(12)r_2,(12)r_1r_2,(12)r_2r_1,(12)r_1r_2r_1\}.$$

To denote the action of (12) on η^*, we take $\alpha=k_1\alpha_1+k_2\alpha_2+k_3\alpha_3\in\eta^*$ and have

$$\begin{aligned}(12)r_1(\alpha)&=(12)(-k_1\alpha_1+k_2(\alpha_1+\alpha_2)+k_3(\alpha_2+\alpha_3))\\&=(12)((k_2+k_3-k_1)\alpha_1+k_2\alpha_2+k_3\alpha_3)\\&=(k_2+k_3-k_1)\alpha_2+k_2\alpha_1+k_3\alpha_3.\end{aligned}$$

Lemma 2 *Let* $A=\begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&-2\end{bmatrix}$ *and* $\alpha=k_1\alpha_1+k_2\alpha_2+k_3\alpha_3\in K, k_1,k_2,k_3\in Z_+, k_3\neq 0$. *If* $r_\alpha\in-(12)W$, *then* $k_1k_2\neq 0$.

Proof By Lemma 1 we have $K = \{\alpha_3\}\bigcup\{k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3|p(k_1, k_2, k_3)\}$. The Weyl group of $g(A)$ is $W = \{1, r_1, r_2, r_1r_2, r_2r_1, r_1r_2r_1\}$

1. If $k_1 = k_2 = 0$, then $\alpha = k_3\alpha_3 \in K$ and hence $\alpha = \alpha_3$. Thus $\alpha^\vee = \frac{2}{(\alpha,\alpha)}\nu^{-1}(\alpha) = -1\cdot\nu^{-1}(\alpha_3) = -\alpha_3^\vee$ and $r_\alpha(\alpha_1) = r_{\alpha_3}(\alpha_1) = \alpha_1 - \alpha_3$. We can get $r_\alpha = r_{\alpha_3} \notin -(12)W$ by checking directly, which is a contradiction.

2. If $k_1 = 0$ and $k_2 \neq 0$, then $\alpha = k_2\alpha_2 + k_3\alpha_3 \in K$ and hence $2k_2 \leq k_3, k_2, k_3 = 1, 2, \cdots$. Set $a = k_2^2 - k_2k_3 - k_3^2$. We know $(\alpha,\alpha) = 2(k_2^2 - k_2k_2 - k_3^2) = 2a \leq 2k_2k_3 < 0$ (so $a < 0$) and $\alpha^\vee = \frac{1}{a}(k_2\alpha_2^\vee + k_2\alpha_3^\vee)$. Therefore,

$$\begin{cases} r_\alpha(\alpha_1) = \alpha_1 + \dfrac{1}{a}(k_2 + k_3)(k_2\alpha_2 + k_3\alpha_3), \\ r_\alpha(\alpha_2) = \alpha_2 + \dfrac{1}{a}(k_3 - 2k_2)(k_2\alpha_2 + k_3\alpha_3), \\ r_\alpha(\alpha_3) = \alpha_3 + \dfrac{1}{a}(k_2 + 2k_3)(k_2\alpha_2 + k_3\alpha_3). \end{cases} \tag{1}$$

We assert that $r_\alpha \notin -(12)W$, which is a contradiction. If this assertion is not true then $r_\alpha \in -(12)W$.

a) If $r_\alpha = (-12)r_1$, then $r_\alpha(\alpha_2) = -(12)r_1(\alpha_2) = -(12)(\alpha_1 + \alpha_2) = -\alpha_1 - \alpha_2$. Combined with (1), we have $a\alpha_1 + (2a + (k_3 - 2k_2)k_2)\alpha + (k_3 - 2k_2)k_3\alpha_3 = 0$. Since $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ is linear independent, we get $a = 0$, which is contradictory to that $a < 0$.

b) If $r_\alpha = -(12)r_1r_2$, then $r_\alpha(\alpha_1) = -(12)r_1r_2(\alpha_1) = -(12)(\alpha_2) = -\alpha_1$. Combined with (1), we obtain $2a\alpha_1 + (k_2 + k_3)k_2\alpha_2 + (k_2 + k_3)k_3\alpha_3 = 0$, which is contrary to that $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ is linear independent.

c) If $r_\alpha \in -(12)W\backslash\{-(12)r_1, -(12)r_1r_2\}$, we can also get the similar contradiction by the same discussion as in a) and b).

3. If $k_1 \neq 0, k_2 = 0$, then $\alpha = k_1\alpha_1 + k_3\alpha_3$. Similar proof as in 2 proves the assertion that $r_\alpha \notin -(12)W$, which is contradictory. Thus Lemma 2 is true.

Theorem 2 *Let* $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & -2 \end{bmatrix}$ *and* $\alpha = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 \in K \subseteq \Delta_+^{\mathrm{im}}$. *Then* $r_\alpha \in -(12)W$ *if and only if* $k_1 = k_2 = k_3 \in Z_+\backslash\{0\}$.

Proof We first prove the "if" part. Let $k_1 = k_2 = k_3 \in Z_+\backslash\{0\}$. Then

$$\alpha = k_1(\alpha_1 + \alpha_2 + \alpha_3),$$
$$(\alpha,\alpha) = k_1^2\sum_{i,j=1}^{3}(\alpha_i, \alpha_j) = -4k_1^2 < 0,$$
$$\alpha^\vee = \frac{2}{(\alpha,\alpha)}\nu^{-1}(\alpha) = -\frac{1}{2k_1}(\alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee).$$

Thus $r_\alpha(\alpha_1) = \alpha_1 - \langle\alpha_1, \alpha^\wedge\rangle\alpha = \alpha_1 + \frac{1}{2k_1}\langle\alpha_1, \alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee\rangle\alpha = \alpha_1$.

On the other hand, $r_1r_2r_1(\alpha_1) = -\alpha_2$. So $-(12)r_1r_2r_1(\alpha_1) = (12)(\alpha_2) = \alpha_1$.

Therefore, $r_\alpha(\alpha_1) = -(12)r_1r_2r_1(\alpha_1)$.

We can also get $r_\alpha(\alpha_2) = -(12)r_1r_2r_1(\alpha_2)$ and $r_\alpha(\alpha_3) = -(12)r_1r_2r_1(\alpha_3)$ by directly checking. Since $\det A = -12$, we have $\dim\eta^* = 3$ and $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ is a basis of η^*. Thus $r_\alpha = -(12)r_1r_2r_1 \in -(12)W$ as a refelction on η^*.

We prove now the "only if" part. If $\alpha = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 \in K \subseteq \triangle_+^{\text{im}}$, we know that $k_1, k_2, k_3 \in Z_+\backslash\{0\}$ and $2k_1 \le k_2 + k_3, 2k_2 \le k_1 + k_3$ and hence $k_1 \le k_3$ and $k_2 \le k_3$ and $k_2 \le k_3$. Let $a = k_1^2 + k_2^2 - k_3^2 - k_1k_2 - k_1k_3 - k_2k_3$. Then

$$\begin{aligned}(\alpha, \alpha) &= \sum_{ij=1} k_ik_j(\alpha_i, \alpha_j) = 2(k_1^2 + k_2^2 - k_3^2 - k_1k_2 - k_1k_3 - k_2k_3) \\ &= 2a \le 2(k_1^2 + k_2^2 - k_3^2 - k_1k_2 - k_1^2 - k_2^2) = -2(k_3^2 + k_1k_2) < 0.\end{aligned}$$

So $a < 0$. It is not difficult to see that

$$\begin{aligned}&\alpha^\vee = \frac{1}{a}(k_1\alpha_1^\vee + k_2\alpha_2^\vee + k_3a_3^\vee), \\ &\langle\alpha_1, \alpha^\wedge\rangle = 2k_1 - k_2 - k_3, \langle\alpha_2, \alpha^\wedge\rangle = 2k_2 - k_1 - k_3 \\ &\langle\alpha_3, \alpha^\wedge\rangle = -k_1 - k_2 - 2k_3.\end{aligned}$$

Thus

$$r_\alpha(\alpha_1) = \alpha_1 - \frac{1}{a}(2k_1 - k_2 - k_3)(k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3), \tag{2}$$

$$r_\alpha(\alpha_2) = \alpha_2 - \frac{1}{a}(2k_2 - k_1 - k_3)(k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3), \tag{3}$$

$$r_\alpha(\alpha_3) = \alpha_3 + \frac{1}{a}(k_1 + k_2 + 2k_3)(k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3). \tag{4}$$

We assert that if $r_\alpha \in -(12)W$ then $r_\alpha = -(12)r_1r_2r_1$. If this is not ture, for example, $r_\alpha = -(12)r_1$, then $r_\alpha(\alpha_1) = -(12)(-\alpha_1) = \alpha_2$. From (2) we have

$$(a - (2k_1 - k_2 - k_3)k_1)\alpha_1 - (a + (2k_1 - k_2 - k_3)k_2)\alpha_2 - (2k_1 - k_2 - k_3)k_3\alpha_3 = 0$$

and get

$$\begin{cases} a - (2k_1 - k_2 - k_3)k_1 = 0, \\ a + (2k_1 - k_2 - k_3)k_2 = 0, \\ (2k_1 - k_2 - k_3)k_3) = 0. \end{cases}$$

Because $k_3 \ne 0$, we obtain $2k_1 - k_2 - k_3 = 0$ and hence $a = 0$, which is a contradiction. Similarly, we can prove that if $r_\alpha = -(12)r_2$, or $r_\alpha = -(12)r_1r_2$, or $r_\alpha = -(12)r_1r_2$, or $r_\alpha = -(12)r_2r_1$, or $r_\alpha = -(12)\cdot 1$ then we get a contradiction as well.

So we must have $r_\alpha = -(12)r_1r_2r_1$ and

$$r_\alpha(\alpha_1) = -(12)r_1r_2r_1(\alpha_1) = -(12)(-\alpha_2) = \alpha_1, \tag{5}$$

$$r_\alpha(\alpha_2) = -(12)r_1r_2r_1(\alpha_2) = -(12)(-\alpha_1) = \alpha_2. \tag{6}$$

From (2), (3), (5) and (6) we obtain that $(2k_1 - k_2 - k_3)\alpha = (2k_2 - k_1 - k_3)\alpha = 0$. Therefore $k_1 = k_2 = k_3$, completing the proof of the Theorem 2.

By the Theorem 2 above, we obtain the following corollary.

Corollary 1 *Let* $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & -2 \end{bmatrix}$ *and* $\alpha = k(\alpha_1 + \alpha_2 + \alpha_3), k = 1, 2, \cdots$. *Then* α *is a special imaginary root of* $g(A)$.

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References

[1] Borcherds R. *Generalized Kac-Moody algebra* [J]. J. Alg., 1988, **115**: 501–512.

[2] Kac V. *Infinite Dimensional Lie Algebras* [M]. 3rd ed. Cambridge University Press, 1990.

[3] Bennett C. *Imaginary roots of a Kac-moody Lie algebra whose reflections preserve root multipeicities* [J]. J. Alg., 1993, **158**: 244–267.

[4] Li Zhenheng. *Some properties of modules over generalized Kac-Moody algebras* [J]. Chinese Science Bullein, 1994, **39**(1): 9–13.

一类特殊广义 Kac-Moody 代数虚根决定的反射与 Weyl 群间的关系

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摘 要: 对有限型李代数 $g(A)$, 相应于每个根 α 的反射 r_α 均在 $g(A)$ 的 Weyl 群 W 中. 当 $g(A)$ 为可对称化的不定型 Kac-Moody 代数时，若 α 为一虚根且 $(\alpha, \alpha) < 0$, 则亦可定义反射 r_α 并有 $r_\alpha \in -W$ 或 r_α 是 $-W$ 中元与一个图自同构之积 (见 [3]). 本文给出了一类秩为 3 的广义 Kac-Moody 代数的虚根系，然后讨论了一类特殊的广义 Kac-Moody 代数的虚根决定的反射与 Weyl 群之间的关系.