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Linear Operators Strongly Preserving M-P Inverses of Matrices over Some Antinegative Commutative Semirings *

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Abstract: Let $\mathcal{S}$ be an antinegative commutative semiring having no zero divisions or finite general Boolean Algebra and $\mu_n(\mathcal{S})$ the set of $n \times n$ matrices over $\mathcal{S}$. In this paper we characterize the structure of the semigroup $\aleph_n(\mathcal{S})$ of linear operators on $\mu_n(\mathcal{S})$ that strongly preserve the M–P inverses of matrices.

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0. Introduction

In recent years, linear preserver problem over various algebraic structrues, including rings, fields and semirings, has been of interest to many authors (see [1]–[6]). With the development of the computer science, linear preserver problem over semirings has special important value. In this paper we character linear operators strongly preserving M–P inverses of matrices over many antinegative commutative semirings $\mathcal{S}$, including the two–element Boolean algebra $\mathcal{B}$, the chain semirings $\mathcal{A}$, the antinegative reals $\mathcal{R}^+$, the antinegative rationals $\mathcal{Q}^+$, the antinegative integers $\mathcal{Z}^+$ and the Boolean algebras $\mathcal{B}_k$ of subsets of a k–element set.

Definitions and properties are given in Section 1. The results for the two–element Boolean algebra matrices are in Section 2, those for nonnegetive semirings having no zero–divisors and for the finite general Boolean algebras are in Sections 3 and 4 respectively.

1. Definitions and properties

Definition 1[1] *A semiring is a binary system* $(\mathcal{H}, +, \times)$ *such that* $(\mathcal{H}, +)$ *is an Abelian monoid(identity 0),* $(\mathcal{H}, \times)$ *is a monoid(identity 1),* $\times$ *distributes over* $+$, $0 \times h = h \times 0 = 0$ *for all* $h \in \mathcal{H}$, *and* $1 \neq 0$.

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Definition 2 *A semiring $\mathcal{H}$ is called antinegative commutative if 0 is the only element to have an additive inverse and $(\mathcal{H}, \times)$ is abelian.*

Obviously, all rings with unity are semirings but no such ring is antinegative. Algebraic items such that unit and zero–divisor of semirings, and linearty and invertibility of linear opertors are as in rings.

Let $\mathcal{S}$ be an antinegative commutative semiring, and let $\mu_n(\mathcal{S})$ denotes the set of all $n \times n$ matrices over $\mathcal{S}$. We denote by O_n, I_n and J_n the zero matrix, the identity matrix and the the matrix of all entries 1 in $\mu_n(\mathcal{S})$ respectively.

For $X, Y \in \mu_n(\mathcal{B})$, we say $Y \geq X$ or $X \leq Y$ if $y_{ij} = 0$ implies $x_{ij} = 0$ for all i, j. We define $X \backslash Y$ to be the matrix $Z = (z_{ij}) \in \mu_n(\mathcal{B})$ such that $z_{ij} = 1$ if and only if $x_{ij} = 1$ and $y_{ij} = 0$ for all i, j.

The number of nonzero entries in a matrix A is denoted $\mid A \mid$. A zero-one matrix are called a cell if $\mid A \mid = 1$. If the nonzero entry occurs in row i and column j, we denote the cell by E_{ij}. When $i \neq j$, we say E_{ij} is an off–diagonal cell; E_{ii} is a diagonal cell. Two cells are collinear if they are in the same row or column.

We denote the Hadamard product of A and B in $\mu_n(\mathcal{S})$ by $A \circ B$. That is $C = A \circ B$ if and only if $c_{ij} = a_{ij} b_{ij}$ for all i and j. For $M \in \mu_n(\mathcal{S})$, the scaling operators $\mathbf{L}_M$, induced by M, is defined by $\mathbf{L}_M : A \to M \circ A$.

We denote by A^+ the M–P inverse of $A \in \mu_n(\mathcal{S})$ which is the solution of the equations:

$$AXA = A, \quad XAX = X, \quad (AX)^{\mathrm{T}} = AX, \quad (XA)^{\mathrm{T}} = XA,$$

where A^{T} the transpose of A. Obviously, if A^+ exists, then it is unique. When $\mathcal{S} = \mathcal{B}$, the following property is showed in [8].

Property 1 *Suppose $X \in \mu_n(\mathcal{B})$. If X^+ exists, then $X^+ = X^{\mathrm{T}}$.*

We say that a linear operator $\mathbf{L}$ on $\mu_n(\mathcal{S})$ strongly preserves M–P inverses of matrices if $[\mathbf{L}(A)]^+$ exists and $[\mathbf{L}(A)]^+ = \mathbf{L}(A^+)$ if and only if $A \in \mu_n(\mathcal{S})$ has a M–P inverse A^+. We denote by $\aleph_n(\mathcal{S})$ the semigroup of all linear operators that preserve M–P inverses of matrices. If $\mathbf{L} \in \aleph_n(\mathcal{S})$, then

Property 2 $\mathbf{L}(C) \neq \bigcirc_n$ *for every cell C.*

Proof Suppose $\mathbf{L}(D) = \bigcirc_n$ for some cell D. Then $\mathbf{L}(D)^+ = \mathbf{L}(D)$, and hence $D^+ = D$. Further D is a diagonal cell. Let F be a cell which is distinct to D and collinear to D. Then $\mathbf{L}(F + D) = \mathbf{L}(F) = \mathbf{L}(F^{\mathrm{T}})^+$, and hence, $D + F = (F^{\mathrm{T}})^+$, an imposibility.

Definition 3 *We say that a linear operator $\mathbf{L} : X \longrightarrow PXP^{-1}$ on $\mu_n(\mathcal{S})$ is permutation similarity if and only if P is a permutation.*

2. The result on two-element Boolean algebra

In this section, we assume that $\mathcal{S} = \mathcal{B}$ and $\mathbf{L} \in \aleph_n(\mathcal{B})$.

Lemma 2.1 *If C is a cell, then $\mathbf{L}(C)$ is a cell.*

Proof It follows from Property 2 that $\mathbf{L}(C) \neq O_n$. Suppose $\mathbf{L}(C) \geq F + G$ for some F and G and $\mathbf{L}(J_n) = F + G + W_1 + \cdots + W_k$, where F, G and W_i $(i = 1, 2, \cdots, k)$

are multually distinct. Then $k \leq n^2 - 2$, and hence $| \mathbf{L}(J_n) \setminus \mathbf{L}(C) | \leq k \leq n^2 - 2$. Let $\mathbf{L}(D_i) \geq W_i$ $(i = 1, 2, \cdots k)$ and $D = C + D_1 + \cdots + D_k$. Then $\mathbf{L}(D) \geq \mathbf{L}(J_n)$. On the other hand, $\mathbf{L}(D) \leq \mathbf{L}(J_n)$ is obvious. Hence $\mathbf{L}(D) = \mathbf{L}(J_n)$. Since $| D | \leq k + 1 \leq n^2 - 1$, we can assume $W \leq (J_n \setminus D)$ for some cell W. Thus $\mathbf{L}(J_n \setminus W) = \mathbf{L}(J_n)$. Again applying $J_n^+ = J_n$, we have $\mathbf{L}(J_n)^+ = \mathbf{L}(J_n)$, and hence $\mathbf{L}(J_n \setminus W) = \mathbf{L}(J_n)^+$. From which, it follows that $J_n^+ = J_n \setminus W$, an imposibility.

Lemma 2.2 $\mathbf{L}$ *is a bijective from the set of all cells to itself.*

Proof From Lemma 2.1, we only need to prove that $\mathbf{L}(F) \neq \mathbf{L}(G)$ for any distinct F and G. Suppose $\mathbf{L}(F) = \mathbf{L}(G)$. Then $\mathbf{L}(F^T) = \mathbf{L}(F)^+ = \mathbf{L}(G)^+$ from Property 1, and thus $F^T = G^+ = G^T$. Hence $F = G$, am impossibility.

Lemma 2.3 *There exists a permutation matrix* $P \in \mu_n(\mathcal{B})$ *such that*

$$P\mathbf{L}(E_{ii})P^{-1} = E_{ii}, \quad \forall i.$$

Proof Suppose C is a diagonal cell. Then $C^+ = C$, and hence $\mathbf{L}(C)^+ = \mathbf{L}(C)$. Applying Lemmas 2.1 and 2.2, we can prove the lemma.

Lemma 2.4 $\mathbf{L}(C)^T = \mathbf{L}(C^T)$ *for any cell* C.

Proof It is obvious from Lemma 2.1 and Property 1.

Theorem 2.1 *The semigroup* $\aleph_n(\mathcal{B})$ *is generated by transposition and the permutation similarty operators.*

Proof When $n = 2$, the theorem follows from Lemmas 2.2 and 2.3. When $n \geq 3$, it follows from Lemma 2.3 that there exists a permutation matrix $P \in \mu_n(\mathcal{B})$ such that

$$P\mathbf{L}(E_{ii})P^{-1} = E_{ii}, \quad \forall i.$$

For any distinct i and j, we can assume that $P\mathbf{L}(E_{ij})P^{-1} = E_{mk}$ and $P\mathbf{L}(E_{ji})P^{-1} = E_{km}$ for some distinct m and k from Lemmas 2.2 and 2.4.

Suppose $m \neq i, j$. Let $A = E_{ij} + E_{ji} + E_{mm}$. Then $\mathbf{L}(A)^+ = \mathbf{L}(A)$ from $A^+ = A$, and hence $(E_{mk} + E_{km} + E_{mm})^3 = (E_{mk} + E_{km} + E_{mm})$. By a direct computation, we have $1 = 0$, an impossibility. Hence $m = i$ or j. Similarly, $k = i$ or j. Noting $m \neq k$, we have $(m, k) = (i, j)$ or $(m, k) = (j, i)$.

Now we prove that $P\mathbf{L}(E_{ij})P^{-1}$ and $P\mathbf{L}(E_{ik})P^{-1}$ are collinear for any multually distinct i, j and k. Otherwise, without loss of generality, let $P\mathbf{L}(E_{ij})P^{-1} = E_{ij}$ and $P\mathbf{L}(E_{ik})P^{-1} = E_{ki}$. Then $P\mathbf{L}(E_{ji})P^{-1} = E_{ji}$ and $P\mathbf{L}(E_{ki})P^{-1} = E_{ik}$ from Lemma 2.4. Let $V = E_{ii} + E_{ij} + E_{ik}$. Then $\mathbf{L}(V^T) = \mathbf{L}(V)^+$ from $V^T = V^+$, and hence $\mathbf{L}(V) = \mathbf{L}(V)\mathbf{L}(V^T)\mathbf{L}(V)$. Comparing the correspond entries of two sides, we have $1 = 0$, an impossibility. The theorem follows from the arbitrarity of i, j and k.

3. Results on some antinegative commutative semirings hving no zero divisors

In this section, we assume that $\mathcal{S}$ is a antinegative commutative semiring having no zero divisors.

For $A = (a_{ij}) \in \mu_n(\mathcal{S})$, we write $\overline{A} = (\psi(a_{ij})) \in \mu_n(\mathcal{B})$, where $\psi$is a map from $\mathcal{S}$ to $\mathcal{B}$ defined by

$$\psi(a) = \begin{cases} 1 & \text{if} \quad a \neq 0 \\ 0 & \text{if} \quad a = 0 \end{cases}, \quad \forall a \in \mathcal{S}.$$

Lemma 3.1[4] *If* $\mathbf{L} : \mu_n(\mathcal{S}) \to \mu_n(\mathcal{S})$ *is a linear operator, let* $\overline{\mathbf{L}}$ *be the operator on* $\mu_n(\mathcal{B})$ *defined by* $\overline{\mathbf{L}} : E_{ij} = \overline{\mathbf{L}(E_{ij})}, \forall i, j$ *for all* i *and* j. *Then* $\overline{\mathbf{L}(A)} = \overline{\mathbf{L}}(\overline{A}), \forall A \in \mu_n(\mathcal{S})$.

Remark 1 If $\mathcal{S}$ have zero divisors(i.e., $ab = 0$ for some $a, b \in S$), let $\mathbf{T}$ be a linear operator on $\mu_n(\mathcal{S})$ defined by

$$\mathbf{T}(E_{ij}) = \begin{cases} bE_{11}, & i = j = 1 \\ O_n, & \text{otherwise} \end{cases}, \quad \forall i, j.$$

Choosing $A = aE_{11}$, we have $\overline{\mathbf{T}(A)} = \bigcirc_n \neq E_{11} = \overline{\mathbf{T}}(\overline{A})$, and hence the restraction of $\mathcal{S}$ has not zero divisors is necessary in Lemma 3.1.

Theorem 3.1 *The semigroup* $\aleph_n(\mathcal{S})$ *is generated by transpose, the permutation similarty operators and the scale operator* $\mathbf{L}_M$, *where* $m_{ij}m_{ji} = 1$ *for all* i *and* j.

Proof Suppose $\mathbf{L} \in \aleph_n(\mathcal{S})$. Then there exist a permutation matrix P such that for any i and j (i) $\overline{\mathbf{L}}(E_{ij}) = PE_{ij}P^{-1}$ or (ii) $\overline{\mathbf{L}}(E_{ij}) = PE_{ji}P^{-1}$ holds from Theorem 2.1.

Suppose (i) holds. Then $\overline{\mathbf{L}(E_{ij})} = \overline{\mathbf{L}}(E_{ij}) = PE_{ij}P^{-1}$ from Lemma 3.1. Thus $\mathbf{L}(E_{ij}) = m_{ij}PE_{ij}P^{-1}$ for some $m_{ij} \in \mathcal{S}$. Applying $E_{ij}^{+} = E_{ji}$, we obtain $\mathbf{L}(E_{ij})^{+} = \mathbf{L}(E_{ji})$, and hence

$$(m_{ij}PE_{ij}P^{-1})(m_{ji}PE_{ji}P^{-1})(m_{ij}PE_{ij}P^{-1}) = m_{ij}PE_{ij}P^{-1}.$$

Comparing the correspond elements of two sides, we have $m_{ij}^2 m_{ji} = m_{ij}$, and hence $\mathbf{L}(E_{ji})^{+} = \mathbf{L}(E_{ij}) = \mathbf{L}(m_{ij}m_{ji}E_{ij})$. Further $E_{ji}^{+} = m_{ij}m_{ji}E_{ij}$. Since E_{ji}^{+} is unique, it follows that $m_{ij}m_{ji} = 1$. Let $M = (m_{ij})$. Then $\mathbf{L}$ is generated by the permutation similarty operators and the scale operator $\mathbf{L}_M$ and $m_{ij}m_{ji} = 1$ for all i and j.

When (ii) holds. The proof is similar.

Remark 2 The inverse of Theorem 3.1 is error. e.g., Let $\mathcal{S} = \mathcal{R}^{+}$, $M = \begin{pmatrix} 1 & 2 \\ \frac{1}{2} & 1 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{pmatrix}$. Then $A^{+} = B$. But $\mathbf{L}_{\mathbf{M}}(A)^{+} \neq \mathbf{L}_{\mathbf{M}}(B)$ (i.e., $\mathbf{L}_M \overline{\in} \aleph_n(\mathcal{R}^{+})$).

Corollary 3.1 *If* 1 *is unique unit in* $\mathcal{S}$, *then the semigroup* $\aleph_n(\mathcal{S})$ *is generated by transposition and the permutation similarty operators.*

Remark 3 1 is unique unit in $\mathcal{A}$ (including the fuzzy semiring and $\mathcal{Z}^{+}$).

Remark 4 If every number of $\mathcal{S}$ is idempotent, then 1 is unique unit in $\mathcal{S}$. In fact, for every unit $a \in \mathcal{S}$, we have $aa^{-1} = 1$, and hence $aa^{-1}a^{-1} = a^{-1}$. Applying a^{-1} idempotent, we have $aa^{-1} = a^{-1}$. Thus $a = 1$.

Let $\mathcal{P}$ is a subring with 1 of the real number ring $\mathcal{R}$. We denote by $\mathcal{P}^+$ the set $\{x \in \mathcal{P} | x \geq 0\}$. Obviously, $\mathcal{P}^+$ is an antinegative commutative semiring having no zero divisors.

Lemma 3.2 *Suppose k is a prime number. Then $k^3 = a^2 + b^2 + c^2$ for some integers a, b and c with $a, b > 0$.*

Proof If $k = 2$, the lemma follows by letting $a = b = 1$ and $c = 0$; If $k \neq 2$, then $k = 1, 3 (\text{mod } 4)$. Thus $k^3 = 1, 3 (\text{mod } 8)$ and k^3 can not be dividen by 4. From [9, TH4], we have $k^3 = a^2 + b^2 + c^2$, where a, b, c are integers. Obviously, nonzero elements in $\{a, b, c\}$ have at least two. Without loss of generality, we assume $a, b > 0$.

Theorem 3.2 *If there exists a positive integer k ($k > 1$) such that k is an unit in $\mathcal{P}^+$ and $n \geq 3$, then the semigroup $\aleph_n(\mathcal{P}^+)$ is generated by transposition and the permutation similarty operators.*

Proof From Theorem 3.1, we only need to prove that $\mathbf{L}_M = \mathbf{L}_{J_n}$.

i) Suppose k is prime. Then $k^3 = a^2 + b^2 + c^2$ for some integers a, b, c with $a, b > 0$ from Lemma 3.2. Let $G = aE_{ij} + bE_{ij} + cE_{im}$. Then $\mathbf{L}_M(G)^+ = (k^3)^{-1}\mathbf{L}_M(G^{\mathrm{T}})$ from $G^+ = (k^3)^{-1}G^{\mathrm{T}}$, and hence $[(k^3)^{-1}(M \circ G^{\mathrm{T}})(M \circ G)]^{\mathrm{T}} = (k^3)^{-1}(M \circ G^{\mathrm{T}})(M \circ G)$. By a direct computation we have $m_{ij} = m_{ji}$. Again applying $m_{ij}m_{ji} = 1$, we obtain $m_{ij} = m_{ji} = 1$ (i.e., $\mathbf{L}_M = \mathbf{L}_{J_n}$).

ii) If k is not prime, then $k = k_1 \cdots k_p$, where $k_1, \cdots, k_p$ are prime. Obvioysly, k_1 is an unit in $\mathcal{P}^+$. This is the form i).

Theorem 3.3 *Let k ($k > 1$) is as in Theorem 3.2, and k is even or can be dividen by $4m+1$ for some integer m. Then $\aleph_2(\mathcal{P}^+)$ is generated by transposition and the permutation similarty operators.*

Proof From Theorem 3.1, we only need to prove that $\mathbf{L}_M = \mathbf{L}_{J_2}$.

i) Suppose $4m + 1 | k$ for some integer m and $k_1 = 4m + 1$. Then $k_1 = a^2 + b^2$ for some positive integers a and b from [7, pp. 127, TH3]. Let $N = aE_{ii} + bE_{ij}$. Then $\mathbf{L}_M(N)^+ = k_1^{-1}\mathbf{L}_M(N^{\mathrm{T}})$ from $N^+ = k_1^{-1}N^{\mathrm{T}}$, and hence, $[k_1^{-1}(M \circ N^{\mathrm{T}})(M \circ N)]^T = k_1^{-1}(M \circ N^{\mathrm{T}})(M \circ N)$. By a direct computation, we have $m_{ij} = m_{ji}$. It follows from $m_{ij}m_{ji} = 1$ that $\mathbf{L}_M = \mathbf{L}_{J_2}$.

ii) Suppose k is even. Then 2 is an unit in $\mathcal{P}^+$. The conclusion follows by choosing $k_1 = 2$ and $a = b = 1$ in the proof of i).

Corollary 3.2 *The semigroup $\aleph_n(\mathcal{R}^+)(\aleph_n(\mathcal{Q}^+))$ is generated by transposition and the permutation similarty operators.*

Proof It is obvious from Theorems 3.2 and 3.3.

Remark 5 Let $\mathcal{P}_n^+ = \{\frac{m}{n^k} | m, k \text{ are antinegative integers}\}$ $(n > 1)$. Then $\mathcal{P}_n^+$ is an antinegative commutative semiring having no zero divisors and satisfies the requirement in Theorem 3.2. Moreover, if n is even or $4m + 1 | n$ for some integer m, then $\mathcal{P}_n^+$ satisfies the requirement in Theorem 3.3.

4. The results on the finite generated Boolean algebras

Let $\mathcal{B}_k$ be the Boolean algebra of subsets of a k–element set Δ_k and let $\sigma_1, \sigma_2, \cdots, \sigma_k$ denote the singleton subsets of Δ_k. We write + for union and denote intersection by juxtaposition. Let 1 and 0 be Δ_k and $\emptyset$ respectively. Obviously, $\mathcal{B}_k$ is an antinegative commutative semiring with zero divisors.

For $A \in \mu_n(\mathcal{B}_k)$ and $1 \le i \le k$, we write $A_i = \sigma_i A$. Obviously,

$$(A+B)_i = A_i + B_i, \quad (AB)_i = A_i B_i, \quad (\alpha A)_i = \alpha_i A_i, \quad (A^{\mathrm{T}})_i = (A_i)^{\mathrm{T}},$$

for any $A, B \in \mu_n(\mathcal{B}_k)$, $\alpha \in \mathcal{B}_k$, and $1 \le i \le k$. From which, we have

Lemma 4.1 *If $A \in \mu_n(\mathcal{B}_k)$, then $A^+ = B$ if and only if A_i^+ exist and $A_i^+ = B_i$ for all $1 \le i \le k$.*

For $1 \le i \le k$, we denote by β_i the Boolean algebra of subsets of σ_i. Obviously, β_i and $\mathcal{B}$ are isophism. From Theorem 2.1, we have

Lemma 4.2 *The semigroup $\aleph_n(\beta_i)$, $\forall 1 \le i \le k$ is generated by transposition and the permutation similarty operators.*

For $\mathbf{L} \in \aleph_n(\mathcal{B}_k)$, let

$$\mathbf{L}_i(X_i) = \sigma_i \mathbf{L}(X), \quad \forall X \in \mu_n(\mathcal{B}_k), \quad 1 \le i \le k.$$

Then $\mathbf{L}_i \in \aleph_n(\beta_i)$ and

$$\mathbf{L}(X) = \sum_{i=1}^{k} \sigma_i \mathbf{L}(X) = \sum_{i=1}^{k} \mathbf{L}_i(X_i), \quad \forall X \in \mu_n(\mathcal{B}_k).$$

Lemma 4.3 $\mathbf{L} \in \aleph_n(\mathcal{B}_k)$ *if and only if* $\mathbf{L}_i \in \aleph_n(\beta_i)$, $\forall 1 \le i \le k$.

Proof The "if" part is obvious. Now we prove the "only if" part.

For fixed $1 \le i \le k$, we choosing $X_i,\ Y_i \in \mu_n(\beta_i)$ with $X_i^+ = Y_i$. Let $Y_j = X_j = I_n$ for all $j \ne i$, $X = \sum_{i=1}^{k} \sigma_i X_i$ and $Y = \sum_{i=1}^{k} \sigma_i Y_i$. Then $X^+ = Y$ from $I_n^+ = I_n$ and Lemma 4.1, and hence $\mathbf{L}(X)^+ = \mathbf{L}(Y)$. Applying lemma 4.1, we have $\mathbf{L}_i(X_i)^+ = \mathbf{L}_i(Y_i)$.

For $1 \le i \le k$, linear operators $\Theta^{(i)}$ and $\Phi^{(i)}$ on $\mu_n(\mathcal{B}_k)$ defined by

$$\Theta^{(i)} : X \longrightarrow \sigma_i X^T + \sigma_i^c X, \quad \forall X \in \mu_n(\mathcal{B}_k)$$

and

$$\Phi^{(i)} : X \longrightarrow \sigma_i P_i X P_i^{-1} + \sigma_i^c X, \quad \forall X \in \mu_n(\mathcal{B}_k)$$

respectively, where P_i is a permutation matrix in $\mu_n(\mathcal{B}_k)$, σ_i^c is the complement of σ_i in $\triangle_k$. We call $\Theta^{(i)}$ the ith rotation operator and $\Phi^{(i)}$ the ith resemblance operator. Obviously, $\Theta^{(i)}, \Phi^{(i)} \in \aleph_n(\mathcal{B}_k)$, $\forall 1 \le i \le k$ and transposition and the permutation similarty operators are generated by the rotation and the resemblance. Again applying Lemmas 4.2 and 4.3, we have

Theorem 4.1 *The semigroup $\aleph_n(\mathcal{B}_k)$ is generated by the rotation and the resemblance.*

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某些非负交换半环上强保持矩阵 M–P 逆的线性算子

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摘要: 设 S 是无零因子的非负交换半环或者有限生成布尔代数，$\mu_n(S)$ 记 S 上的矩阵集合. 本文确定了 $\mu_n(S)$ 上的强保持矩阵 M–P 逆的线性算子半群 $\aleph_n(S)$ 的结构.