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Graded FS-Rings *

CHEN Jian-hua
(Dept. of Math., Teachers' College, Yangzhou Univ., Jiangsu 225002)

Abstract: In this paper, we introduce the notation of graded FS-module of the graded module over a group G-graded ring and obtain some characterization involving graded maximal graded left ideal for a graded FS-ring and some equivalent conditions between the ring R and group ring RG, the graded ring R and group ring of graded ring $R[G]$.

Key words: graded ring; graded FS-ring; graded FS-module; group ring of (graded) ring.

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1. Notations and Preliminaries

In this paper, all rings have a unitary and all modules are unitary. If R is a ring, by an R-module we will mean a left R-module, and we will denote the category of R-modules by R-mod. If G is a group and $R = \oplus_{\sigma\in G}R_\sigma$ is a graded ring of type G, the category of graded left R-modules will be denoted by R-gr. If $M = \oplus_{\sigma\in G} M_\sigma$, $N = \oplus_{\sigma\in G}N_\sigma \in R$-gr, Hom_{R-gr}(M,N) is the set morphisms in the category R-gr from M to N, i.e. $\mathrm{Hom}_{R-gr}(M,N) = \{f : M \to N | f$ is R-linear and $f(M_\sigma) \subseteq N_\sigma, \forall\sigma \in G\}$

If $R = \oplus_{\sigma\in G}R_\sigma$ is a graded ring, we say that R is a strongly graded ring if $R_\sigma R_\tau = R_{\sigma\tau}$ for any $\sigma,\tau \in G$. It is well known [1] that R is a strongly graded ring if and only if $R_\sigma R_{\sigma^{-1}} = R_e$ for any $\sigma \in G$. (e is the identity of the group G). If R is graded by a finite group G, the smash product, $R\#G^*$, is a free right and left R-module with basis $\{P_\sigma|\sigma \in G\}$ and multiplication determined by $(rP_\sigma)(sP_\tau) = rs_{\sigma\tau^{-1}}P_\tau$, where $s_{\sigma\tau^{-1}}$ is the $\sigma\tau^{-1}$ component of s.

Let R be a (graded) ring. The left and right annihilators of a subset X of R are written $l(X)$ and $r(X)$ respectively. The (graded) socle of a (graded) left R-module M is written (gr-Soc(M)) Soc(M).

We recall that a study of rings with flat left socle and be called left FS-rings by Liu Zhongkui[2]. Some equivalent conditions of such rings are given in [2], In the same paper,

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Biography: CHEN Jianhua (1963-), male, born in Rugao county, Jiangsu province. M.Sc.

it was proved that R is a left FS-ring if and only if S is a left FS-ring, where S is a excellent extension of R.

In this paper we consider graded rings with graded flat graded left socle. We call such graded rings to be graded left FS-rings. Some equivalent conditions of such graded rings are given in section 2 in terms of either the graded maximal graded left ideals of R or graded left R-modules with graded flat graded socle. In section 3 we consider group rings of ring R and group rings of graded ring R with (graded) flat (graded) left socle. We show that if R is a ring, G is finite group and $|G|^{-1} \in R$, then R is a FS-ring if and only if RG is a graded FS-ring, if R is a graded ring, G is a finite group and $|G|^{-1} \in R$, then R is a (graded) FS-ring if and only if $R[G]$ is a (graded) FS-ring.

2. Properties and Characterizations

By analogy with left FS-modules and left FS-rings, we give the following definiton.

Definition 1 *A graded left R-module M is called a graded FS-module if every graded simple submodule is graded flat.*

Definition 2 *A graded ring R is called a graded left (right) FS-ring if ${}_RR(R_R)$ is a graded FS-module.*

Lemma 2.1[1] *Let $R = \oplus_{\sigma \in G} R_\sigma$ be a strongly graded ring. Then the functor $R \otimes_{Re} : R_e$-mod→ R - gr given by $M \to R \otimes_{R_e} M$ where $M \in R_e$-mod and $R \otimes_{R_e} M$ is a graded R-module by the grading $(R \otimes_{R_e} M)_\sigma = R_\sigma \otimes_{Re} M$, is an equivalence, Its inverse is the functor $(\cdot)_e : R$- gr→ R_e-mod given by $M \to M_e$ where $M \in R$-gr and $M = \oplus_{\sigma \in G} M_\sigma$.*

Theorem 2.2 *Let $R = \oplus_{\sigma \in G} R_\sigma$ is a stronly G-graded ring, $M = \oplus_{\sigma \in G} M_\sigma \in R$-gr. Then M is a graded FS-module if and only if M is an FS-R_e-module. In particular, R is a graded FS-ring if and only if R_e is an FS-ring.*

Proof Let M be an FS-R_e-module. If $N = \oplus_{\sigma \in G} N_\sigma$ is a graded simple submodule of M, then N_σ=0 or N_σ is a simple R_e-module. Thus N_σ is a flat R_e-module by hypothesis, in particular, N_e is a flat R_e-module. By Lemma 2.1 N is a graded flat module, and so M is a graded flat module. Similarly the necessity can be proved by Lemma 2.1.

It is clear the graded PS-modules, (see[4]), are graded FS-modules, and graded left PS-rings are graded left FS-rings.

Example 1 Let R be a graded von-Neumann reguler ring. Then R is a graded FS-ring.

Example 2 Let R be a FS-ring. Then the group ring RG is a graded FS-ring by theorem 2.2.

Corollary 2.3 *Let R be a strongly graded ring, $|G|^{-1} \in R$, then the following are equivalent.*

(1) R_e is an FS-ring.
(2) R is a gr-FS-ring.
(3) $R\#G^*$ is an FS-ring.

In order to give characterizations of graded left FS-rings, now we give following lemmas.

Lemma 2.4 *Let R be a G-graded ring, $B \in gr\text{-}R$ and an exact sequence $0 \to K \xrightarrow{i} F \xrightarrow{f} B \to 0$, in which F is gr-flat. then B is graded flat if and only if $K \cap IF = IK$ for each graded right ideal I of R.*

Proof Because the functor $\otimes_R$ is right exact, so we have the commutative diagram

$$\begin{array}{ccccccc} & I \otimes_R K & \xrightarrow{1 \otimes i} & I \otimes_R F & \xrightarrow{1 \otimes f} & I \otimes_R B & \to 0 \\ & \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 & \\ 0 \to & IK & \xrightarrow{i} & IF & \xrightarrow{f} & IB & \end{array}$$

Where α_1, α_2 and α_3 are clear. As B is a graded flat and $0 \to I \xrightarrow{i} R$, so $0 \to I \otimes_R B \to R \otimes_R B \simeq B$. It follows that α_3 is an isomorphism. so is α_2. Since α_1 is surjective, therefore $0 \to IK \to IF \to IB \to 0$ is a exact sequence. and

$$IF/IK \cong IF + K/K \cong IF/IF \cap K.$$

It is easy to see $IK = IF \cap K$. The proof of sufficiency is similar to without grading.

A graded module F is called a graded free if F has a basis of homogeneous elements, or equivalently $F \cong \oplus_{\sigma \in S} R(\sigma)$, where S is a subset of G.

Lemma 2.5 *Let R be a G-graded ring, $B \in$ gr-R and an exact sequence $0 \to K \xrightarrow{i} F \xrightarrow{f} B \to 0$. In which F is graded free. Suppose $\{x_j | j \in \Gamma, x_j \in h(F)\}$ is a basic of F, If $v \in h(k)$, say $v = r_1 x_{j_1} + r_2 x_{j_2} + \cdots + r_t x_{j_t} (v_i \in h(R)), I(v)$ denote the graded right ideal generate by $r_1, r_2 \cdots r_t$, then B is graded flat if and only if $v \in I(v)K$ for all $v \in K$.*

Proof Let B be a graded flat module. Then $IF \cap K = IK$ by lemma 2.4. If $v = r_1 x_{j_1} + r_2 r_{j_2} + \cdots + r_t x_{j_t} \in K$ and $\{r_1, r_2, \cdots, r_t\} \subseteq I(v)$, so $v \in I(v)F$, Since $I(v) = r_1 R + r_2 R + \cdots + r_t R, I(v)K = r_1 RK + r_2 RK + \cdots + r_t RK = r_1 K + r_2 K + \cdots + r_t K$, therefore there exist $y_1, y_2 \cdots y_t \in h(K)$ such that $v = r_1 y_1 + r_2 y_2 + \cdots + r_t y_t$. Thus $v \in I(v)K$.

To the contrary, suppose $v \in I(v)K$ for all $v \in K$, let I be a graded right ideal of R. It is easy to see, $IK \subseteq K \cap IF$. If $v \in K \cap IF$, say $v = s_1 x_{j_1} + s_2 x_{j_2} + \cdots + s_n x_{j_n}$, where $s_1, s_2, \cdots s_n \in I$ and $I(v) \subseteq I$, so $v \in IK$. Thus $K \cap IF = IF$. B is a graded flat module by Lemma 2.4.

Lemma 2.6 *Let R be a G-graded ring, I be a graded left ideal of R. Then I is graded direct summand if and only if there exist $f \in R_e, f^2 = f$, such that $I = Rf$.*

The next result gives several characterizations of graded left FS-rings in terms of graded left FS-modules, or graded maximal graded left ideals, or all graded simple graded left R-modules.

Theorem 2.7 *The following are equivalent for a graded ring R by group G.*

(1) R is a graded left FS-ring.

(2) Gr-Soc $({}_R R)$ is graded flat.

(3) R has a faithful graded left FS-module.

(4) If L is a maximal graded left ideal of R then either $r(L) = 0$ or $a \in aL$ for every $a \in L \cap h(R)$.

(5) If L is a graded essential maximal graded left ideal of R then either $r(L) = 0$ or $a \in aL$ for every $a \in L \cap h(R)$.

(6) Every graded simple graded left R-module ${}_RM$ is either graded flat or $\mathrm{Hom}_{R-gr}(M, R) = 0$.

Proof The equivalence of (1) and (4) is trival.

(1)$\Leftrightarrow$ (2) By proposition 1.2.18 ([1]), a graded left R module is graded flat if and only if M is flat. We have $\oplus_{\alpha \in \Gamma} M_\alpha$ is graded flat if and only if M_α is flat for all $\alpha \in \Gamma$.

(1)$\Rightarrow$ (3) It is clear.

(3)$\Rightarrow$ (4) By analogy with the proof of Theorem of [4], let ${}_RM$ be a faithful graded FS-module and let L be a maximal graded left ideal. If $r(L) \neq 0$ write $T = r(L)$ so that $LT = 0$. On the other hand $RT \neq 0$, so $RTM \neq 0$ by hypothesis, say $Rm_\sigma \neq 0$ where $m_\sigma \in TM \cap h(M)$ and $\deg(m_\sigma) = \sigma$. Thus $L \subseteq L(m_\sigma) \neq R$, so $L = l(m_\sigma)$. Definition:

$$g : R \to Rm_\sigma$$

$$r_\sigma \to r_\tau m_\sigma$$

then $g \in \mathrm{Hom}(R, Rm_\sigma)_\sigma$, i.e. $g \in \mathrm{Hom}_{R-gr}(R, Rm_\sigma(\sigma^{-1}))$ and $R/L \cong Rm_\sigma(\sigma^{-1})$. Then R/L is graded flat by hypothesis and Lemma 2.5. Thus $a \in aL$ for every $a \in L \cap h(R)$.

(4)$\Rightarrow$ (5) It is clear.

(5)$\Rightarrow$ (1) Let $a \in R_\sigma$ and Ra be a minimum graded left ideal of R. Definition:

$$g : R \to Ra$$

$$r_\tau \to r_\tau a$$

then $g \in \mathrm{Hom}(R, Ra)_\sigma$, that is $g \in \mathrm{Hom}_{R-gr}(R, Ra(\sigma^{-1}))$, so $Ra \cong R/L$ in gr-R, where L=L(a) is a maximal graded left ideal of R. If L is not graded essential then $L = Rf$ for some $f^2 = f \in R_e$ by Lemma 2.6, thus $Ra(\sigma^{-1}) \cong R(1-f)$, which implies that $Ra(\sigma^{-1})$ is graded projective. Now, suppose that L is a graded essential maximal graded left ideal. Assume that $r(L) \neq 0$. Then $a \in aL$ for every $a \in L \cap h(R)$, Thus $R/L \cong Ra(\sigma^{-1})$ is a graded flat left R-module. Now suppose that $r(L) = 0$. Set $h \in \mathrm{Hom}_{R-gr}(Ra, R)$ if $(a)h = b \in R_\sigma$, then $Lb = L((a)h) = (La)h = (0)h = 0$, it follows that $a = 0$. So we have proved that every graded minimum graded left ideal of R is flat.

Corollary 2.8 *Suppose that R is a graded left FS-module. Then, (1) for every graded essential maximal graded left ideal L of R, either* $1(L) = 0$ *or $r(L) = 0$.*

(2) for every maximal graded left ideal L of R, either $l(L) = 0$ or $r(L) = fR$ where $f^2 = f \in R_e$.

Proof (1) Let L be a graded essential maximal graded left ideal. Assume that $l(L) \neq 0$, the $L \cap l(L) \neq 0$. Choose $0 \neq a \in L \cap l(L)$, if $v(L) \neq 0$, then, by Theorem 2.7 $b \in bL$ for every $b \in L$. Thus $a \in aL$. which implies $a = 0$, a contradiction. Therefore we have $r(L) = 0$.

(2) By Lemma 2.6.

3. Group Rings RG and R[G]

In this section we consider group rings with (graded) flat left socle. Let R be a ring and G be a finite group, as is known to all, group ring RG is strongly graded ring of type G. First of all, we have the following.

Theorem 3.1 *Let R be a graded ring by finite group $G.|G|^{-1} \in R$, then the following are equivalent.*

(1) R is an FS-ring.

(2) RG is an FS-ring.

(3) RG is a gr-FS-ring.

Proof (1)$\Leftrightarrow$ (2) By Corollary 3.5[2].

(1)$\Leftrightarrow$ (3) By Theorem 2.2.

By Remark 3.3 [2] and [4] we have following.

Corollary 3.2 *Let R be a graded ring by finite group $G.|G|^{-1} \in R$, then the following are equivalent.*

(1) R is a PS-ring.

(2) RG is a PS-ring.

(3) RG is a graded PS-ring.

According to C. Nastasescu[6], if $R = \oplus_{\sigma \in G} R_\sigma$ is a graded ring of type G, he denote by $R[G]$ the left free R-module with the basis $\{\sigma | \sigma \in G\}$, i.e. $R[G] = \{\sum_{g \in G} \lambda_g g | \lambda_g \in R\}$. For the elements $\lambda_\sigma \tau$ and $\lambda_{\sigma'} \tau'$ where $\lambda_\sigma \in R_\sigma, \lambda_{\sigma'} \in R_{\sigma'}$, he define their product by

$$(\lambda_\sigma \tau)(\lambda_{\sigma'} \tau') = \lambda_\sigma \lambda_{\sigma'} (\sigma'^{-1} \tau \sigma' \tau').$$

$R[G]$ is called the group rings of graded rings. If define for every $\sigma \in G, (R[G])_\sigma = \sum_{\lambda\mu=\sigma} R_\lambda \mu = \sum_{\tau \in G} R_{\sigma\tau^{-1}} \tau = \oplus_{\tau \in G} R_{\sigma\tau^{-1}} \tau$ the $R[G]$ is a G-graded ring, and $(R[G])_e = \sum_{\sigma \in G} R_{\sigma^{-1}} \sigma$. The following lemma appeared in [6].

Lemma 3.3 *With the above notations, we have*

(1) $R[G]$ is a strongly graded ring with the grading $\{(R[G])_\sigma, \sigma \in G\}$.

(2) $\varphi : R \to (R[G])_e, \varphi(\sum_{\sigma \in G} \lambda_\sigma) = \sum_{\sigma \in G} \lambda_\sigma \sigma^{-1}$, where $\lambda_\sigma \in R$, is a ring isomorphism.

(3) If I is a graded left ideal of R, then $I[G]$ is a graded left ideal of $R[G]$ and $I[G] \cap (R[G])_e = \varphi(I)$.

Theorem 3.4 *Let R be a graded ring by finite group G, $|G|^{-1} \in R$. Then*

(1) R is an FS-ring if and only if $R[G]$ is an FS-ring.

(2) R is an FS-ring if and only if $R[G]$ is an gr-FS-ring.

(3) If R is strongly graded, R is a gr-FS-ring if and only if $R[G]$ is a gr-FS-ring.

Proof (1) In fact $R[G]$ is a crossed product (see[1]). By [2] corollary 3.5 it is clear.

(2) By lemma 3.3, $\varphi : R \to (R[G])_e, \varphi(\sum_{\sigma \in G} \lambda_\sigma) = \oplus_{\sigma \in G} \lambda_\sigma \sigma^{-1}$ is a ring isomorphism. On the other hand, it is easy to see that every $g \in G$ commute with any element of $(R[G])_e$

and therefore $R[G]$ is the group ring $(R[G])_e$ by the group G in the classical sense. Thus the result follows from Theorem 2.2 and Lemma 3.3.

(3) Since $(R[G])_e = \oplus_{\sigma\in G}(R_\sigma \sigma^{-1}), (R_\tau = \tau^{-1}) \subseteq R_{\sigma\tau}(\sigma\tau)^{-1}$, so $(R[G])_e$ is a graded ring of type G. Thus φ is a graded ring isomorphism. If R is strongly graded, then R is an FS-ring if and only if R is a gr-FS-ring.

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陈 建 华

(扬州大学师范学院数学系，江苏 225002)

摘 要: 本文引进群分次环上分次模的分次 FS- 模的概念，利用分次极大分次左理想给出分次 FS- 环的几个刻画，得到了环 R 和群环 RG, 分次环 R 和分次环的群环 $R[G]$ 间的几个等价条件.