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On the Transfinite Interpolation and Approximation by a Class of Periodic Bivariate Cubic Splines on Type-II Triangulation *

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Abstract: In this paper, we discuss the transfinite interpolation and approximation by a class of periodic bivariate cubic Splines on type-II triangulated partition $\Delta_{mn}^{(2)}$. the existence, uniqueness and the expression of interpolation periodic bivariate splines are given. And at last, we estimate their approximation order.

Key words: transfinite interpolation; periodic bivariate cubic splines; type-II triangulation; approximation order.

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1. Introduction

Recently, there are many papers studying the Periodic Bivariate Spline interpolation on rectangular [1]–[8], It is obvious that the methods in [1]–[2] are significant, but we must solve many multidimensional equations and must give some higher partial derivative values in interpolation conditions.

In this paper, we will give a new method, by which the Periodic Bivariate Cubic spline interpolations can be constructed conveniently.

For convenience, we follow the notations in [2]–[4]. In [2], it was shown that if one chooses the following 16 parameters in rectangle $D_{ij}: w_5, y_5, y_6, x_6, x_5, t_5, t_6, w_6, w_1, w_3, u_1$, $S(p,q)(p=i,i+1; q=j,j+1)$ and $S(i+\frac{1}{2}, j+\frac{1}{2})$ (Fig.1) then there exists uniquely a piecewise bivariate cubic spline $S_{ij}(x,y) \in C'(D_{ij})$.

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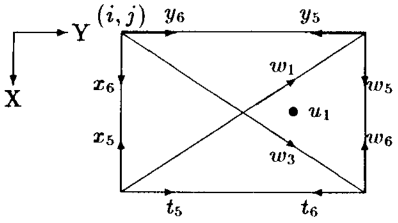
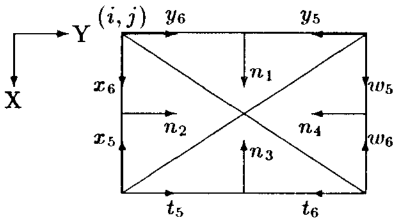


(Fig.1)



(Fig.2)

It is obvious that w_1, w_3, u_1 and $S(i+\frac{1}{2}, j+\frac{1}{2})$ are the interior parameters of D_{ij}. In this paper, at first, we point out that the 4 interior parameters can be replaced by the normal derivative values on the midpoints of four sides of D_{ij},that is, if we give the following 16 parameters: w_5, w_6, y_5, y_6, x_5, x_6, t_5, t_6, n_i $(i = 1, 2, 3, 4)$ and $S(p, q)$ $(p = i, i+1; q = j, j+1)$ (Fig.2) then there exists uniquely a piecewise bivariate cubic spline $S_{ij}(x, y) \in C^1(D_{ij})$, where

$$n_1 = \frac{h_1}{2}\frac{\partial S}{\partial x}(i, j+\frac{1}{2}), n_4 = (-\frac{h_2}{2})\frac{\partial S}{\partial y}(i+\frac{1}{2}, j+1),$$

$$n_3 = (-\frac{h_1}{2})\frac{\partial S}{\partial x}(i+1, j+\frac{1}{2}), n_2 = \frac{h_2}{2}\frac{\partial S}{\partial y}(i+\frac{1}{2}, j).$$

Secondly, since we choose n_i $(i = 1, 2, 3, 4)$ as parameters, then the piecewise bivariate cubic splines satisfy the C^1 continuous conditions on nets of D, so that we can construct various C^1-interpolation problems on D conveniently. In this case, one does not have to solve multidimensional equations and does not need higher partial derivative values. Hence,the method in this paper is clearly an improvement to [1]–[4]. In fact, we give a new piecewise cubic spline finite elements, (Fig.2) which is also better than the bicubic spline finite elements introduced in [5].

2. The existence and uniqueness

Let D denote the rectangular domain $[0, l_1] \otimes [0, l_2]$, trianglated by a so-called type-II partition $\Delta_{mn}^{(2)}$ (Fig.3). The spaces $\overline{S}_3^1(\Delta_{mn}^{(2)})$ of double periodic bivariate spline functions defined on D are that If $S(x, y) \in \overline{S}_3^1(\Delta_{mn}^{(2)})$, then $S(x, y)$ satisfy the following[1]:

(i) $S(x, y) \in C^1(D)$;

(ii) S coincides with a bivariate polynomials of degree at most 3 on each triangle of D;

(iii) $S(0, y) = S(l_1, y)$, $S(x, 0) = S(x, l_2)$

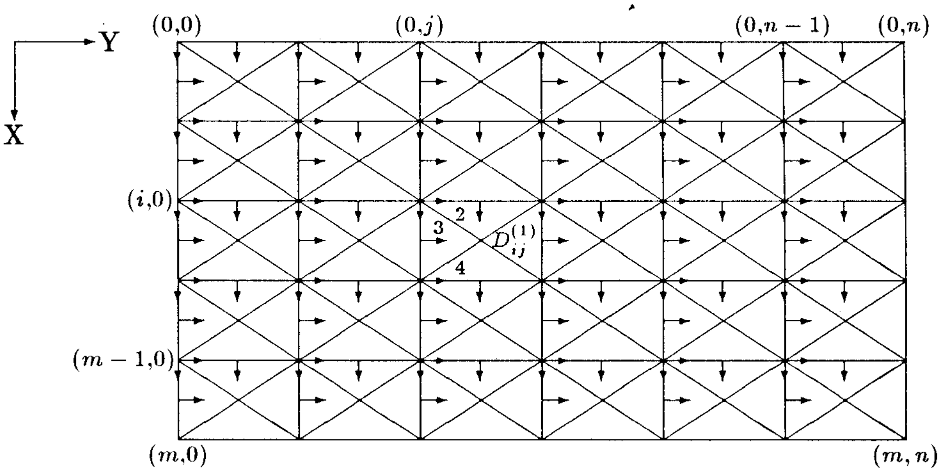
$$\frac{\partial}{\partial x}S(0, y) = \frac{\partial}{\partial x}S(l_1, y)(0 \le y \le l_2), \quad \frac{\partial}{\partial x}S(x, 0) = \frac{\partial}{\partial x}S(x, l_2)(0 \le x \le l_1),$$

$$\frac{\partial}{\partial y}S(0, y) = \frac{\partial}{\partial y}S(l_1, y)(0 \le y \le l_2), \quad \frac{\partial}{\partial y}S(x, 0) = \frac{\partial}{\partial y}S(x, l_2)(0 \le x \le l_1),$$

Theorem 2.1 *Let $f(x, y)$ be the double periodic derivative function in [2], then there is a unique double periodic bivariate cubic spline $S(x, y) \in \overline{S}_3^1(\Delta_{mn}^{(2)})$, which satisfies following interpolation conditions:*

$$(i)\ (S-f)_{(i,j)} = 0, \qquad (i=0,1,\cdots,m-1;\quad j=0,1,\cdots,n-1)$$
$$(ii)\ \frac{\partial}{\partial x}(S-f)_{(i,j)} = 0, \frac{\partial}{\partial y}(S-f)_{(i,j)} = 0, \quad (i=0,1,\cdots,m-1;\quad j=0,1,\cdots,n-1)$$
$$(iii)\ \frac{\partial}{\partial x}(S-f)_{(i,j+\frac{1}{2})} = 0, \qquad (i=0,1,\cdots,m-1;\quad j=0,1,\cdots,n-1)$$
$$\frac{\partial}{\partial y}(S-f)_{(i+\frac{1}{2},j)} = 0, \qquad (i=0,1,\cdots,m-1;\quad j=0,1,\cdots,n-1)$$

It is obvious that all the interpolation conditions are given on the nets of D (Fig.3). Hence, it is a transfinite interpolation [6].



(Fig.3)

Proof Let a bivariate cubic spline S(x,y) be defined on a triangular cell $D_{ij}^{(1)}$ $(0 \le i \le m-2,\ 0 \le j \le n-2)$, then by barycentric coordinates it can be expressed as [2]:

$$S(x,y) = S(\alpha,\beta,\gamma) = [(a_i);\alpha,\beta,\gamma],$$

and

$$\frac{\partial S}{\partial \alpha} = 3a_1\alpha^2 + 2a_4\alpha\beta + a_5\beta^2 + 2a_6\alpha\gamma + a_7\gamma^2 + a_{10}\beta\gamma,$$
$$\frac{\partial S}{\partial \beta} = 3a_2\beta^2 + a_4\alpha^2 + 2a_5\alpha\beta + 2a_8\beta\gamma + a_9\gamma^2 + a_{10}\alpha\gamma,$$
$$\frac{\partial S}{\partial \gamma} = 3a_3\gamma^2 + a_6\alpha^2 + 2a_7\alpha\beta + a_8\beta^2 + 2a_9\beta\gamma + a_{10}\alpha\beta.$$

Let $\Delta = \frac{1}{4}h_1h_2$, then

$$\begin{aligned}\frac{\partial S}{\partial y}\Big|_{(i+\frac{1}{2},j+1)} &= \frac{h_1}{4\Delta}\left[\frac{\partial S}{\partial \beta} + \frac{\partial S}{\partial \gamma} - 2\frac{\partial S}{\partial \alpha}\right]_{(0,\frac{1}{2},\frac{1}{2})} \\ &= \frac{h_1}{4\Delta}\left[\frac{1}{4}(3a_2+2a_3+a_9) + \frac{1}{4}(3a_3+2a_9+a_8) - \frac{1}{2}(a_5+a_7+a_{10})\right].\end{aligned}$$

If we set $\frac{\partial}{\partial y}(S-f)_{(i+\frac{1}{2},j+1)}=0$, then

$$3(a_2+a_3+a_8+a_9)-2(a_5+a_7+a_{10})=4h_2\frac{\partial f}{\partial y}(i+\frac{1}{2},j+1),$$

then by [2;(2.1)], we have

$$-2(6S(i+\frac{1}{2},j+\frac{1}{2})+2w_1+2w_3+u_1)=F_1(i,j), \tag{2.1}$$

where

$$F_1(i,j):=4h_2\frac{\partial f}{\partial y}(i+\frac{1}{2},j+1)-6f(i,j+1)-6f(i+1,j+1)-2\frac{\partial f}{\partial x}(i,j+1)h_1+ \\ 2\frac{\partial f}{\partial x}(i+1,j+1)h_1-\frac{\partial f}{\partial y}(i,j+1)h_2-\frac{\partial f}{\partial y}(i+1,j+1)h_2. \tag{2.2}$$

Similarly, in other cells $D_{ij}^{(k)}(k=2,3,4)$ we have

$$\begin{aligned}-2(6S(i+\frac{1}{2},j+\frac{1}{2})+2w_1-2w_3-u_1)&=F_2(i,j),\\ -2(6S(i+\frac{1}{2},j+\frac{1}{2})+2w_1-2w_3+u_1)&=F_3(i,j),\\ -2(6S(i+\frac{1}{2},j+\frac{1}{2})-2w_1+2w_3-u_1)&=F_4(i,j),\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}F_2(i,j):=&4h_1\frac{\partial f}{\partial x}(i,j+\frac{1}{2})-6f(i,j)-6f(i,j+1)-2\frac{\partial f}{\partial y}(i,j)h_2+\\ &2\frac{\partial f}{\partial y}(i,j+1)h_2+\frac{\partial f}{\partial y}(i+1,j)h_2+\frac{\partial f}{\partial y}(i,j)h_2,\\ F_3(i,j):=&4h_2\frac{\partial f}{\partial y}(i+\frac{1}{2},j)-6f(i+1,j)-6f(i,j)+2\frac{\partial f}{\partial x}(i,j)h_1-\\ &2\frac{\partial f}{\partial x}(i,j)h_1+\frac{\partial f}{\partial y}(i+1,j)h_2+\frac{\partial f}{\partial y}(i,j)h_2,\\ F_4(i,j):=&4h_1\frac{\partial f}{\partial x}(i+1,j+\frac{1}{2})-6f(i+1,j+1)-6f(i+1,j)+\\ &2\frac{\partial f}{\partial y}(i+1,j+1)h_2-2\frac{\partial f}{\partial y}(i+1,j)h_2-\\ &\frac{\partial f}{\partial x}(i+1,j+1)h_1-\frac{\partial f}{\partial x}(i+1,j)h_1.\end{aligned} \tag{2.4}$$

It is obvious that the equations with the unknown variables w_1, w_3, u_1, and $S(i+\frac{1}{2},j+\frac{1}{2})$ formed by (2,1) and (2.2) can be solved as follows:

$$S(i+\frac{1}{2},j+\frac{1}{2})=-\frac{1}{48}\Big\{4h_2\frac{\partial f}{\partial y}(i+\frac{1}{2},j+1)+4h_1\frac{\partial f}{\partial x}(i+1,j+\frac{1}{2})+4h_2\frac{\partial f}{\partial y}(i+\frac{1}{2},j)+ \\ h_1\left[\frac{\partial f}{\partial x}(i+1,j+1)-\frac{\partial f}{\partial x}(i,j+1)+\frac{\partial f}{\partial x}(i+1,j)-\frac{\partial f}{\partial x}(i,j)\right]+4h_1\frac{\partial f}{\partial x}(i,j+\frac{1}{2})- \\ h_2\left[\frac{\partial f}{\partial y}(i,j)+\frac{\partial f}{\partial y}(i+1,j)-\frac{\partial f}{\partial y}(i,j+1)-\frac{\partial f}{\partial y}(i+1,j+1)\right]-$$

$$12\left[f(i+1,j+1)+f(i,j)+f(i,j+1)+f(i+1,j)\right]\Big\},$$

$$\begin{aligned}
w_1 = \frac{1}{16}\Big\{ & h_1\left[\frac{\partial f}{\partial x}(i,j+1)+\frac{\partial f}{\partial x}(i+1,j)-3\frac{\partial f}{\partial x}(i,j)-3\frac{\partial f}{\partial x}(i+1,j+1)\right]-4h_1\frac{\partial f}{\partial x}(i,j+\frac{1}{2})+ \\
& h_2\left[3\frac{\partial f}{\partial y}(i+1,j+1)+3\frac{\partial f}{\partial f}(i,j)-\frac{\partial f}{\partial y}(i+1,j)-\frac{\partial f}{\partial y}(i,j+1)\right]+4h_1\frac{\partial f}{\partial x}(i+1,j+\frac{1}{2})- \\
& 4h_2\frac{\partial f}{\partial y}(i+\frac{1}{2},j+1)+4h_2\frac{\partial f}{\partial y}(i+\frac{1}{2},j)-12\left[f(i+1,j)-f(i,j+1)\right]\Big\}, \\
w_3 = \frac{1}{16}\Big\{ & h_1\left[3\frac{\partial f}{\partial x}(i+1,j)+3\frac{\partial f}{\partial x}(i,j+1)-\frac{\partial f}{\partial x}(i,j)-\frac{\partial f}{\partial x}(i+1,j+1)\right]+4h_1\frac{\partial f}{\partial x}(i,j+\frac{1}{2})+ \\
& h_2\left[3\frac{\partial f}{\partial y}(i+1,j)+3\frac{\partial f}{\partial y}(i,j+1)-\frac{\partial f}{\partial y}(i,j)-\frac{\partial f}{\partial y}(i+1,j+1)\right]-4h_1\frac{\partial f}{\partial x}(i+1,j+\frac{1}{2})- \\
& 4h_2\frac{\partial f}{\partial y}(i+\frac{1}{2},j+1)+4h_2\frac{\partial f}{\partial y}(i+\frac{1}{2},j)+12\left[f(i+1,j+1)-f(i,j)\right]\Big\}, \\
u_1 = -\frac{1}{8}\Big\{ & 3h_1\left[\frac{\partial f}{\partial x}(i+1,j)-\frac{\partial f}{\partial x}(i,j+1)+\frac{\partial f}{\partial x}(i+1,j+1)-\frac{\partial f}{\partial x}(i,j)\right]+4h_2\frac{\partial f}{\partial y}(i+\frac{1}{2},j)+ \\
& 3h_2\left[\frac{\partial f}{\partial y}(i+1,j)+\frac{\partial f}{\partial y}(i,j)-\frac{\partial f}{\partial y}(i,j+1)-\frac{\partial f}{\partial y}(i+1,j+1)\right]+4h_2\frac{\partial f}{\partial y}(i+\frac{1}{2},j+1)- \\
& 4h_1\frac{\partial f}{\partial x}(i,j+\frac{1}{2})-4h_1\frac{\partial f}{\partial x}(i+1,j+\frac{1}{2})\Big\}. \qquad (2.5)
\end{aligned}$$

Thus, using interpolation conditions (i)–(iii) and [2;(2.1)], we can easily give the piecewise spline expression on D_{ij}, for example, on $D_{ij}^{(1)}$, and obtain the coefficients of $s(x,y)$ uniquely as follows:

$$\begin{aligned}
a_1 &= S(i+\frac{1}{2},j+\frac{1}{2}), \\
a_2 &= f(i,j), \\
a_3 &= f(i+1,j+1), \\
a_4 &= 3S(i+\frac{1}{2},j+\frac{1}{2})+w_1, \\
a_5 &= 3f(i,j+1)+\frac{1}{2}[\frac{\partial f}{\partial x}(i,j+1)h_1+\frac{\partial f}{\partial y}(i,j+1)h_2], \\
a_6 &= 3S(i+\frac{1}{2},j+\frac{1}{2})+w_3, \\
a_7 &= 3f(i,j+1)-\frac{1}{2}[\frac{\partial f}{\partial x}(i+1,j+1)h_1+\frac{\partial f}{\partial y}(i+1,j+1)h_2], \\
a_8 &= 3f(i,j+1)+\frac{\partial f}{\partial x}(i,j+1)h_1, \\
a_9 &= 3f(i+1,j+1)-\frac{\partial f}{\partial x}(i+1,j+1)h_1, \\
a_{10} &= 6S(i+\frac{1}{2},j+\frac{1}{2})+2w_1+2w_3+u_1.
\end{aligned}$$

Similarly we can write the expressions on other cells $D_{ij}^{(k)}$ $(k=2,3,4)$ $(0 \le i \le m-2,\ 0 \le j \le n-2)$ symmetrically and we omit their details. Thus, we can get the piecewise bivariate cubic splines on D_{ij} and by [2], they can C^1-continuously be pieced together on two diagonals of D_{ij}. Now, we consider the existence of interpolation splines on D_{0n-1}, from the interpolation condition and periodic conditions and by the method described in the above on $D_{0n-1}^{(1)}$, we have

$$\begin{aligned}
S(\frac{1}{2}, n-\frac{1}{2}) = -\frac{1}{48}\Big\{&(4h_2\frac{\partial f}{\partial y}(\frac{1}{2},n) + 4h_1\frac{\partial f}{\partial x}(1,n-\frac{1}{2}) + 4h_2\frac{\partial f}{\partial y}(\frac{1}{2},n-1) + \\
&h_1\left[\frac{\partial f}{\partial x}(1,0) - \frac{\partial f}{\partial x}(0,0) + \frac{\partial f}{\partial x}(1,n-1) - \frac{\partial f}{\partial x}(0,n-1)\right] + 4h_1\frac{\partial f}{\partial x}(0,n-\frac{1}{2}) - \\
&h_2\left[\frac{\partial f}{\partial y}(0,n-1) + \frac{\partial f}{\partial y}(1,n-1) - \frac{\partial f}{\partial y}(0,0) - \frac{\partial f}{\partial y}(1,0)\right] - \\
&12\left[f(1,0) + f(0,n-1) + f(0,0) + f(1,n-1)\right]\Big\},
\end{aligned}$$

$$\begin{aligned}
w_1 = \frac{1}{16}\Big\{&h_1\left[\frac{\partial f}{\partial x}(0,0) + \frac{\partial f}{\partial x}(1,n-1) - 3\frac{\partial f}{\partial x}(0,n-1) - 3\frac{\partial f}{\partial x}(1,0)\right] + 4h_2\frac{\partial f}{\partial y}(\frac{1}{2},n-1) - \\
&h_2\left[3\frac{\partial f}{\partial y}(1,0) + 3\frac{\partial f}{\partial y}(0,n-1) - \frac{\partial f}{\partial y}(1,n-1) - \frac{\partial f}{\partial y}(0,0)\right] - 4h_2\frac{\partial f}{\partial y}(\frac{1}{2},n) + \\
&4h_1\frac{\partial f}{\partial x}(1,n-\frac{1}{2}) - 4h_1\frac{\partial f}{\partial x}(0,n-\frac{1}{2}) - 12\left[f(1,n-1) - f(0,0)\right]\Big\},
\end{aligned}$$

$$\begin{aligned}
w_3 = \frac{1}{16}\Big\{&h_1\left[3\frac{\partial f}{\partial x}(1,n-1) + 3\frac{\partial f}{\partial x}(0,0) - \frac{\partial f}{\partial x}(0,n-1) - \frac{\partial f}{\partial x}(1,0)\right] + 4h_2\frac{\partial f}{\partial y}(\frac{1}{2},n-1) - \\
&h_2\left[3\frac{\partial f}{\partial y}(1,n-1) + 3\frac{\partial f}{\partial y}(0,0) - \frac{\partial f}{\partial y}(0,n-1) - \frac{\partial f}{\partial y}(1,0)\right] + 4h_1\frac{\partial f}{\partial x}(0,n-\frac{1}{2}) - \\
&4h_1\frac{\partial f}{\partial x}(1,n-\frac{1}{2}) - 4h_2\frac{\partial f}{\partial y}(\frac{1}{2},n) + 12\left[f(1,0) - f(0,n-1)\right]\Big\},
\end{aligned}$$

$$\begin{aligned}
u_1 = -\frac{1}{8}\Big\{&3h_1\left[\frac{\partial f}{\partial x}(1,n-1) - \frac{\partial f}{\partial x}(0,0) + \frac{\partial f}{\partial x}(1,0) - \frac{\partial f}{\partial x}(0,n-1)\right] + \\
&3h_2\left[\frac{\partial f}{\partial y}(1,n-1) + \frac{\partial f}{\partial y}(0,n-1) - \frac{\partial f}{\partial y}(0,0) - \frac{\partial f}{\partial y}(1,0)\right] + 4h_2\frac{\partial f}{\partial y}(\frac{1}{2},n) + \\
&4h_2\frac{\partial f}{\partial y}(\frac{1}{2},n-1) - 4h_1\frac{\partial f}{\partial x}(0,n-\frac{1}{2}) - 4h_1\frac{\partial f}{\partial x}(1,n-\frac{1}{2})\Big\}. \qquad (2.7)
\end{aligned}$$

Thus, we get the coefficients of $S(x,y)$ on $D_{0n-1}^{(1)}$ as follows:

$$\begin{aligned}
a_1 &= S(\frac{1}{2}, n-\frac{1}{2}),\\
a_2 &= f(0,n-1),\\
a_3 &= f(1,0),\\
a_4 &= 3S(\frac{1}{2}, n-\frac{1}{2}) + w_1,\\
a_5 &= 3f(0,0) + \frac{1}{2}[\frac{\partial f}{\partial x}(0,0)h_1 + \frac{\partial f}{\partial y}(0,0)h_2],
\end{aligned}$$

$$a_6 = 3S(\frac{1}{2}, n - \frac{1}{2}) + w_3,$$
$$a_7 = 3f(0,0) - \frac{1}{2}[\frac{\partial f}{\partial x}(1,0)h_1 + \frac{\partial f}{\partial y}(1,0)h_2],$$
$$a_8 = 3f(0,0) + \frac{\partial f}{\partial x}(0,0)h_1,$$
$$a_9 = 3f(1,0) - \frac{\partial f}{\partial x}(1,0)h_1,$$
$$a_{10} = 6S(\frac{1}{2}, n - \frac{1}{2}) + 2w_1 + 2w_3 + u_1.$$

Similarly, we can find the expression on $D^{(k)}_{0n-1}$ $(k = 2,3,4)$. And in the same way, we can obtain the piecewise bivariate cubic splines on $D^{(k)}_{in-1}$ $(k = 1,2,3,4)$ $(1 \le i \le m-1)$ and $D^{(k)}_{m-1j}$ $(k = 1,2,3,4)$ $(0 \le j \le n-1)$ and by [1], they are C^1-continuously pieced together on two diagonals of D_{in-1} and D_{m-1j}, $(0 \le i \le m-1,\ 0 \le j \le n-1)$. and they satisfy the periodic bivariate interpolation conditions. Since we give three directional derivative values of every side of D_{ij} in interpolation conditions (ii) and (iii), then by [7,p206], If we let D_{ij} and D_{ij+1} be two neighboring rectangles, and $S_1(x,y)$, $S_2(x,y)$ are the bivariate cubic splines on D_{ij} and D_{ij+1} respectively, then S_1, S_2 are C^1-joined on public line $\overline{(i,j+1)(i+1,j+1)}$. Similarly, let S_1, S_3 denote the piecewise splines on D_{ij} and D_{ij+1}, then S_1, S_3 are C^1-joined on $\overline{(i,j+1)(i+1,j+1)}$. Thus, along the x-direction and y-direction , beginning with D_{00} and joining each other piece by piece, we can obtain the piecewise spline $S(x,y)$ on D which satisfies the interpolation conditions (i)–(iii) and is C^1-continous on the nets $x = x_i$, $y = y_j (i = 1,2,\cdots,m-1; j = 1,2,\cdots,n-1)$. Our proof is completed.

3. Approximation order

Let $\overline{C}^{(4)}(D)$ denote a periodic function space on D such that

(i) $f(x,y) \in C^4(D)$

(ii) $f(0,y) = f(l_1,y), \qquad f(x,0) = f(x,l_2)$

$$\frac{\partial}{\partial x}f(0,y) = \frac{\partial}{\partial x}f(l_1,y), \quad \frac{\partial}{\partial x}f(x,0) = \frac{\partial}{\partial x}f(x,l_2),$$
$$\frac{\partial}{\partial y}f(0,y) = \frac{\partial}{\partial y}f(l_1,y), \quad \frac{\partial}{\partial y}f(x,0) = \frac{\partial}{\partial y}f(x,l_2).$$

Theorem 3.2 *Let $f(x,y)$ and $S(x,y)$ be the double periodic derivative function, and bivariate cubic interpolation spline as in Theorem 2.2 respectively, if $f \in \overline{C}^{(4)}(D)$ then we have*

$$|f-S| \le K(l)\cdot \max\left(\rho_\Delta, \rho_\Delta^{-1}\right)\left[h^2\|f\|_* + h^2\omega\left(\partial^4 f, h\right)\right], \quad \text{for } (x,y) \in D\backslash[0,l_1-h_1]\otimes[0,l_2-h_2],$$
$$|f-S| \le K(l)\cdot \max\left(\rho_\Delta, \rho_\Delta^{-1}\right)\left[h\|f\|_* + h^2\omega\left(\partial^4 f, h\right)\right], \quad \text{for } (x,y) \in [0,l_1-h_1]\otimes[0,l_2-h_2].$$

Proof For simplicity, we let $y_{ij} = y_6$, $x_{ij} = x_6$, $y_{ij+1} = -y_5$, $x_{ij+1} = w_5$, $x_{i+1j} = -x_5$, $y_{i+1j} = t_5$, $x_{i+1j+1} = -w_6$, $y_{i+1j+1} = -t_6$, $w_{ij} = w_1$, $t_{ij} = w_3$, and $u_{ij} = u_1$ in D_{ij}

(Fig.1). Correspondently, we can write the parameters in D_{ij+1} and D_{i+1j}. Considering the C^1-continuity on $\overline{(i,j+1)(i+1,j+1)}$, by [4,p382], we have

$$u_{ij}+2w_{ij}+2t_{ij}+6S(i+\frac{1}{2},j+\frac{1}{2})+u_{ij+1}-2w_{ij+1}-2t_{ij+1}-6S(i+\frac{1}{2},j+1+\frac{1}{2})$$
$$=-2x_{i+1j+1}+2x_{ij+1}+6S(i,j+1)+6S(i+1,j+1). \tag{3.1}$$

Let us denote by $\bar{S}(x,y)$ the interpolation spline in [3], and correspondently, let $\bar{u}_{ij},\bar{w}_{ij},\bar{t}_{ij}$ etc denote their parameters, then by [3,p14], we have

$$\bar{u}_{ij}+2\bar{w}_{ij}+2\bar{t}_{ij}+6\bar{S}(i+\frac{1}{2},j+\frac{1}{2})+\bar{u}_{ij+1}-2\bar{w}_{ij+1}-2\bar{t}_{ij+1}-6\bar{S}(i+\frac{1}{2},j+1+\frac{1}{2})$$
$$=-2\bar{x}_{i+1j+1}+2\bar{x}_{ij+1}+6\bar{S}(i,j+1)+6\bar{S}(i+1,j+1). \tag{3.2}$$

From (3.2) and (3.1) and by using their interpolation conditions respectively. We have

$$\theta_{ij}+2\alpha_{ij}+2\beta_{ij}+6\gamma_{ij}+\theta_{ij+1}+2\alpha_{ij+1}+2\beta_{ij+1}+6\gamma_{ij+1}=4(\alpha_{ij+1}+\beta_{ij+1}), \tag{3.3}$$

where

$$\theta_{ij}=u_{ij}-\bar{u}_{ij},\alpha_{ij}=w_{ij}-\bar{w}_{ij},\beta_{ij}=t_{ij}-\bar{t}_{ij},$$
$$\gamma_{ij}=S(i+\frac{1}{2},j+\frac{1}{2})-\bar{S}(i+\frac{1}{2},j+\frac{1}{2}).$$

From [3] we know $\overline{w}_{ij}=\frac{\partial f}{\partial x}(i+\frac{1}{2},j+\frac{1}{2})\frac{h_1}{2}+\frac{\partial f}{\partial y}(i+\frac{1}{2},j+\frac{1}{2})\frac{h_2}{2}$, hence from (2.5) we have

$$\begin{aligned}\alpha_{ij+1}=\frac{1}{16}\Big\{&4h_1\frac{\partial f}{\partial x}(i+1,j+\frac{1}{2})-4h_1\frac{\partial f}{\partial x}(i,j+\frac{1}{2})+4h_2\frac{\partial f}{\partial y}(i+\frac{1}{2},j)+\\&h_1\left[\frac{\partial f}{\partial x}(i,j+1)+\frac{\partial f}{\partial x}(i+1,j)-3\frac{\partial f}{\partial x}(i,j)-3h_1\frac{\partial f}{\partial x}(i+1,j+1)\right]+\\&h_2\left[3\frac{\partial f}{\partial y}(i+1,j+1)+3\frac{\partial f}{\partial y}(i,j)-\frac{\partial f}{\partial y}(i+1,j)-\frac{\partial f}{\partial y}(i,j+1)\right]-\\&4h_2\frac{\partial f}{\partial y}(i+\frac{1}{2},j+1)-12[f(i+1,j)-f(i,j+1)]\Big\}-\\&\frac{1}{2}h_1\frac{\partial f}{\partial x}(i+\frac{1}{2},j+\frac{1}{2})-\frac{1}{2}h_2\frac{\partial f}{\partial y}(i+\frac{1}{2},j+\frac{1}{2}).\end{aligned} \tag{3.4}$$

If $f\in C^4(D)$, then we expand the right sides in (3.4) around the point $(i+\frac{1}{2},j+1)$ by Taylor formula and get

$$|\alpha_{ij+1}|\le K(l)h^4\omega(\partial^4 f,h). \tag{3.5}$$

Similarly, we can get

$$|\beta_{ij+1}|\le K(l)h^4\omega(\partial^4 f,h). \tag{3.6}$$

Thus, from (3.3), we have the following recursion relations:(fixed i)

$$|\theta_{ij+1}+2\alpha_{ij+1}+2\beta_{ij+1}+6\gamma_{ij+1}|\le|\theta_{ij}+2\alpha_{ij}+2\beta_{ij}+6\gamma_{ij}|+K(l)h^4\omega(\partial^4 f,h), \tag{3.7}$$

and specially, if $i=0$, then we have recursion relations:

$$|\theta_{0j+1}+2\alpha_{0j+1}+2\beta_{0j+1}+6\gamma_{0j+1}|\le|\theta_{0j}+2\alpha_{0j}+2\beta_{0j}+6\gamma_{0j}|+K(l)h^4\omega(\partial^4 f,h). \tag{3.8}$$

Let $D_0 = \bigcup_{j=0}^{m-1} D_{0j}$ be the first row of rectangular domain D triangulated by type-II triangulation. We estimate the errors in D_0, at first,in $D_{00}^{(1)}$, by [3] and (2.5),(2.6),We have

$$|s - \bar{s}| \le 7|\gamma_{00}| + |\alpha_{00}| + |\beta_{00}| + |6\gamma_{00} + 2\alpha_{00} + 2\beta_{00} + \theta_{00}|. \tag{3.9}$$

From (2.1),(2.3),we have

$$\mathcal{A} \cdot \mathcal{X} = \mathcal{F}, \tag{3.10}$$

where

$$\mathcal{A} = \begin{pmatrix} 6 & 2 & 2 & 1 \\ 6 & 2 & -2 & -1 \\ 6 & -2 & -2 & 1 \\ 6 & -2 & 2 & -1 \end{pmatrix},$$

$$\mathcal{X} = (\gamma_{00}, \alpha_{00}, \beta_{00}, \theta_{00})^T,$$

$$\mathcal{F} = (\widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2, \widetilde{\mathcal{F}}_3, \widetilde{\mathcal{F}}_4)^T,$$

and

$$\begin{aligned} \widetilde{\mathcal{F}}_1 &= F_1(0,0) + 12f(\frac{1}{2}, \frac{1}{2}) + 4w_{00} + 4t_{00} + 2u_{00}, \\ \widetilde{\mathcal{F}}_2 &= F_2(0,0) + 12f(\frac{1}{2}, \frac{1}{2}) - 4w_{00} - 4t_{00} - 2u_{00}, \\ \widetilde{\mathcal{F}}_3 &= F_3(0,0) + 12f(\frac{1}{2}, \frac{1}{2}) - 4w_{00} - 4t_{00} + 2u_{00}, \\ \widetilde{\mathcal{F}}_4 &= F_4(0,0) + 12f(\frac{1}{2}, \frac{1}{2}) - 4w_{00} + 4t_{00} - 2u_{00}. \end{aligned}$$

If $f \in C^4(D)$, then by substituting (2.2) ,(2.4) and (2.5) into the above expression and expanding every term of $\widetilde{\mathcal{F}}_j (j = 1,2,3,4)$ at the point $(\frac{1}{2}, 1)$ by the Taylor formula, we have

$$|\widetilde{\mathcal{F}}_j| \le K(l) h^4 \omega(\partial^4 f, h) \qquad (j = 1,2,3,4). \tag{3.11}$$

Therefore from (3.10), we have

$$|\gamma_{00}|, |\alpha_{00}|, |\beta_{00}|, |\theta_{00}| \le K(l) h^4 \omega(\partial^4 f, h). \tag{3.12}$$

We can get similar estimation as above on $D_{00}^{(k)} (k = 2,3,4)$ and obtain the following estimation on D_{00}:

$$|\theta_{00} - 2\alpha_{00} + 2\beta_{00} + 6\gamma_{00}| \le K(l) h^4 \omega(\partial^4 f, h). \tag{3.13}$$

By the recursion relations (3.8), we have

$$|\theta_{0j+1} + 2\alpha_{0j+1} + 2\beta_{0j+1} + 6\gamma_{0j+1}| \le K(l) h^3 \omega(\partial^4 f, h). \tag{3.14}$$

Since the length of recursion does not exceed n, combining (3.5) and (3.6), we have

$$|\theta_{0j+1} + 6\gamma_{0j+1}| \le K(l) h^3 \omega(\partial^4 f, h). \tag{3.15}$$

and by the definition of $\theta_{0j+1}, \gamma_{0j+1}$ and $\bar{u}_{0j+1}$, we can get

$$\begin{aligned}\theta_{0j+1} - 6\gamma_{0j+1} &= u_{0j+1} - \bar{u}_{0j+1} - 6S(\frac{1}{2}, j+1+\frac{1}{2}) + 6f(\frac{1}{2}, j+1+\frac{1}{2}) \\ &= u_{0j+1} - \left[\frac{\partial^2 f}{\partial x^2}(\frac{1}{2}, j+1+\frac{1}{2})\left(\frac{h_1}{2}\right)^2 + \right. \\ &\left. \frac{\partial^2 f}{\partial y^2}(\frac{1}{2}, j+1+\frac{1}{2})\left(\frac{h_2}{2}\right)^2 + \frac{\partial^2 f}{\partial xy}(\frac{1}{2}, j+1+\frac{1}{2})\frac{h_1}{2}\frac{h_2}{2}\right] - \\ &6S(\frac{1}{2}, j+1+\frac{1}{2}) + 6f(\frac{1}{2}, j+1+\frac{1}{2}).\end{aligned}$$

If $f \in C^4(D)$, using (2.5) and expanding the rights of above on point $(\frac{1}{2}, j+1)$ by Taylor formula, we can get

$$|\theta_{0j+1} - 6\gamma_{0j+1}| \le K(l)h^4\|f\|_*. \tag{3.16}$$

Similarly, we can obtain the estimation as above on $D_{0j}^{(k)}$ (k=2,3,4), thus, from (3.15), (3.16), (3.5) and (3.6),we have following estimations on $D_0 (j = 0, 1, \cdots, n-1)$

$$|\gamma_{0j}|, |\alpha_{0j}|, |\beta_{0j}|, |\theta_{0j}| \le K(l)[h^3\omega(\partial^4 f, h) + h^4\|f\|_*]. \tag{3.17}$$

Since S(x,y) is C^1-continous on $\overline{(i+1,j)(i+1,j+1)}$, by [4,p382], we have

$$\begin{aligned}&-u_{i+1j+1} - 2t_{i+1j+1} + 2w_{i+1j+1} - u_{ij+1} - 2w_{ij+1} + 2t_{ij+1} + 6S(i+\frac{1}{2}, j+1+\frac{1}{2}) + 6S(i+1+\frac{1}{2}, j+1+\frac{1}{2}) \\ &= 2y_{i+1j+1} - 2y_{i+1j+2} + 6S(i+1, j+1) + 6S(i+1, j+2).\end{aligned} \tag{3.18}$$

By the same method, we can get the estimations on first column $D_1 = \bigcup_{i=0}^{m-1} D_{i0}$:

$$|\gamma_{i0}|, |\alpha_{i0}|, |\beta_{i0}|, |\theta_{i0}| \le K(l)[h^3\omega(\partial^4 f, h) + h^4\|f\|_*]. \tag{3.19}$$

Using (3.1) and (3.18), we can get the following recursion relations [4,p382]:

$$\begin{aligned}&(u_{ij} + 2t_{ij}) - (u_{i+1j+1} + 2t_{i+1j+1}) \\ &= 4w_{ij+1} - 2w_{ij} - 2w_{i+1j+1} + 2x_{ij+1} - 2x_{i+1j+1} + 2y_{i+1j+1} - 2y_{i+1j+2} + \\ &\quad 6S(i, j+1) + 6S(i+1, j+1) + 6S(i+1, j+1) + 6S(i+1, j+2) - \\ &\quad 6S(i+\frac{1}{2}, j+\frac{1}{2}) - 6S(i+\frac{1}{2}, j+1+\frac{1}{2}) - 6S(i+\frac{1}{2}, j+1+\frac{1}{2}) - \\ &\quad 6S(i+1+\frac{1}{2}, j+1+\frac{1}{2}).\end{aligned} \tag{3.20}$$

Hence, using the interpolation conditions of S and $\overline{S}$,we have

$$\begin{aligned}(\theta_{ij} + 2\beta_{ij}) - (\theta_{i+1j+1} + 2\beta_{i+1j+1}) &= 4\alpha_{ij+1} - 2\alpha_{ij} - 2\alpha_{i+1j+1} - \\ &\quad 6[\gamma_{ij} + 2\gamma_{ij+1} + \gamma_{i+1j+1}].\end{aligned} \tag{3.21}$$

By (2.5) and by expanding $\gamma_{ij} = S(i+\frac{1}{2}, j+\frac{1}{2}) - f(i+\frac{1}{2}, j+\frac{1}{2})$ at the point $(i+\frac{1}{2}, j+\frac{1}{2})$ by Taylor formula, we have

$$|\gamma_{ij}| \le K(l)\|f\|_* h^4 \quad (j = 1, 2, \cdots, n-1). \tag{3.22}$$

Combining (3.5) and (3.21), we get the recursion relation

$$|\theta_{i+1j+1} + 2\beta_{i+1j+1}| \le |\theta_{ij} + 2\beta_{ij}| + K(l)h^4[\omega(\partial^4 f, h) + \|f\|_*]. \tag{3.23}$$

From (3.17) and (3.19), we have

$$|\theta_{i0} + 2\beta_{i0}|, |\theta_{0j} + 2\beta_{0j}| \le +K(l)[h^3\omega(\partial^4 f, h) + h^4\|f\|_*].$$

Since the length of recursion (3.23) does not exceed max(m,n),we have

$$|\theta_{i+1j+1} + 2\beta_{i+1j+1}| \le K(l)[h^2\omega(\partial^4 f, h) + h^3\|f\|_*]. \tag{3.24}$$

and combining (3.6), we get

$$|\theta_{i+1j+1}| \le K(l)[h^2\omega(\partial^4 f, h) + h^3\|f\|_*]. \tag{3.25}$$

Thus, from (3.5), (3.6), (3.22) and (3.25), we have the following estimation on $D_{ij}(i \neq 0, j \neq 0)$:

$$|\gamma_{ij}|, |\alpha_{ij}|, |\beta_{ij}|, |\theta_{ij}| \le K(l)[h^2\omega(\partial^4 f, h) + h^3\|f\|_*]. \tag{3.26}$$

According to the methods in [2]–[4], for $f \in C^4(D)$, we can get

$$|s - \bar{s}| \le K(l)[h^2\omega(\partial^4 f, h) + h^3\|f\|_*]. \tag{3.27}$$

By [1], if $f \in \overline{C}^{(4)}(D)$, then we have

$$|f - \bar{S}| \le K(l)\cdot\max\left(\rho_\Delta, \rho_\Delta^{-1}\right)\left[h^2\|f\|_* + h^2\omega\left(\partial^4 f, h\right)\right], \text{ for } (x,y) \in D\backslash[0, l_1-h_1]\otimes[0, l_2-h_2],$$

$$|f - \bar{S}| \le K(l)\cdot\max\left(\rho_\Delta, \rho_\Delta^{-1}\right)\left[h\|f\|_* + h^2\omega\left(\partial^4 f, h\right)\right], \text{ for } (x,y) \in [0, l_1-h_1]\otimes[0, l_2-h_2]. \tag{3.28}$$

Combining (3.27) and (3.28), we complete the proof of theorem 3.1.

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一类 II 型剖分下的二元三次周期样条的超限插值和逼近

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摘 要: 本文讨论了 II- 型三角剖分 $\Delta_{mn}^{(2)}$ 下的一类二元三次周期样条的超限插值和逼近，给出了它的表示以及存在唯一性，最后，估计了它的逼近阶.